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Discrete integrable systems and deformations of associative algebras

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Abstract
Interrelations between discrete deformations of the structure constants for associative algebras and discrete integrable systems are reviewed. Theory of deformations for associative algebras is presented. Closed left ideal generated by the elements representing the multiplication table plays a central role in this theory. Deformations of the structure constants are generated by the deformation driving algebra and governed by the central system of equations. It is demonstrated that many discrete equations such as discrete Boussinesq equation, discrete WDVV equation, discrete Schwarzian KP and BKP equations, discrete Hirota–Miwa equations for KP and BKP hierarchies are particular realizations of the central system. An interaction between the theories of discrete integrable systems and discrete deformations of associative algebras is reciprocal and fruitful. An interpretation of the Menelaus relation (discrete Schwarzian KP equation), discrete Hirota–Miwa equation for KP hierarchy, consistency around the cube as the associativity conditions and the concept of gauge equivalence, for instance, between the Menelaus and KP configurations are particular examples.

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1. Introduction

Theory of solitons and modern theory of deformations of associative algebras have almost the same age, a little above 40. Geographically, they were born pretty near to each other: one in Princeton University [1, 2] and the other in Pennsylvania University [3, 4]. Moreover, both these theories have used one common principal concept, namely the concept of deformation. An idea of isospectral deformations was one of the first basic ideas in the theory of integrable equations [5]. Then, the dressing method [6], bi-Hamiltonian structures [7] as well as the

Backlund and Darboux transformations represent particular realizations of the various classes of deformations for solutions of solitons equations (see e.g. [8, 9] and references therein). On the other hand, one of the approaches to the deformation theory of associative algebras proposed in [4] was '... to take the point of view that the objects being deformed are not merely algebras, but essentially algebra with a fixed basis' and to treat 'the algebraic set of all structure constants as parameter space for deformation theory'.

In spite of this sharing of the idea of deformation, the theory of integrable equations and deformation theory for associative algebras for the first 25 years have been developed independently, without any interconnection and influence of one to another. An apparent difference between the basic objects in these theories, i.e. between the dependent variables in the nonlinear equations and structure constants of associative algebras, seemed to be so big that, it was thought, these theories cannot have anything in common.

The situation has changed drastically in the beginning of the 1990s with the discovery of Witten [10] and Dijkgraaf–Verlinde–Verlinde [11] (WDVV). They demonstrated that the function F which defines the correlation function $\langle \Phi_j \Phi_k \Phi_l \rangle = \frac{\partial^3 F}{\partial x^j \partial x^k \partial x^l}$ etc in the perturbed two-dimensional topological field theory obeys the system of equations

$$\sum_{s,t=1}^{N} \frac{\partial^3 F}{\partial x^j \partial x^k \partial x^s} g^{st} \frac{\partial^3 F}{\partial x^t \partial x^m \partial x^l} = \sum_{s,t=1}^{N} \frac{\partial^3 F}{\partial x^l \partial x^k \partial x^s} g^{st} \frac{\partial^3 F}{\partial x^t \partial x^m \partial x j} \tag{1}$$

where $j, k, l, m = 1, 2, \ldots, N$ and g^{st} are constants. In terms of C^l_{jk} defined as

$$C^l_{jk} = \sum_{m=1}^{N} g^{lm} \frac{\partial^3 F}{\partial x^j \partial x^k \partial x^m}, \tag{2}$$

the WDVV equation (1) is of the form

$$\sum_{m=1}^{N} C^m_{jk}(x) C^n_{ml}(x) = \sum_{m=1}^{N} C^m_{kl}(x) C^n_{jm}(x), \tag{3}$$

which is nothing but the condition of associativity for the structure constants C^l_{jk} of the N-dimensional algebra of primary fields Φ_j [10, 11]. Thus, each solution $F(x)$ of the WDVV equation describes a deformation of the structure constants by formula (2).

This result has provided us with the remarkable realization of Gerstenhaber's approach mentioned above. On the other hand, it has revealed a striking connection between the theory of deformations of associative algebras and nonlinear partial differential equations (PDEs).

WDVV equation (1) and formulae (2, 3) have been immediately interpreted and formalized by Dubrovin [12, 13] as the theory of Frobenius manifolds. It provides us with the classes of deformations of the so-called Frobenius algebras. An extension of this approach to general algebras and corresponding F-manifolds has been given in [14]. It was shown within these theories that not only the WDVV equation but also many other integrable systems of nonlinear PDEs both dispersionless and dispersive describe deformations of associative algebras (see e.g. [13, 15–18]). In few years, the interconnection between the theories of Frobenius and F-manifolds on one side and the theory of integrable nonlinear PDEs on the other has been well established.

An alternative approach to deformations of structure constants for associative algebras proposed recently in the papers [19–24] has allowed us to construct wider classes of deformations. They include the coisotropic [19, 20], quantum [21], discrete [22, 23] deformations as well as general deformations generated by the so-called deformation driving algebra (DDA) [24]. These classes of deformations are governed by dispersionless, dispersive, discrete and difference integrable systems with some well-known integrable equations among them. One of the characteristic features of the method developed in [19–24] is that it allows us to construct different classes of deformations, for instance, coisotropic, quantum and discrete deformations of the same algebra just choosing different DDAs.

Theory of integrable systems is nowadays a well-developed and rich theory which includes a vast variety of nonlinear ordinary and partial differential equations and discrete equations (see e.g. [25–28]). Theory of discrete integrable systems forms a very important branch of the whole theory. Toda lattice [29] was the first such equation studied by the inverse scattering method [30, 31]. After that several methods to construct and solve integrable differential-difference and discrete equations have been developed (see e.g. [32–40]). Soon, it became clear that some of the discrete integrable equations play a fundamental role in the whole theory: they are the generating equations for the infinite hierarchies of continuous integrable equations and encode the basic algebraic structures associated with these hierarchies (see e.g. [35–41]). Discrete integrable equations have also served as the basic tool in the formulation and study of discrete geometry (see [42–44]) and discrete complex analysis [45–47]. Moreover, some of discrete equations are directly connected with basic theorems, such as the Menelaus theorem, of the classical geometry [48–51]. Due to their significance discrete integrable systems certainly merit a profound study from all possible viewpoints.

Our principal goal here is to discuss recent results on the interrelations between some basic discrete integrable systems and discrete deformations of structure constants of associative algebras. We will present first a general theory of deformations of associative algebras generated by the Lie algebra DDA and governed by the so-called central system (CS) of equations. We will show how discrete Boussinesq and WDVV equations, discrete Schwarzian Kadomtsev–Petviashvili (DSKP) equation, discrete bilinear Hirota–Miwa equations for Kadomtsev–Petviashvili (KP) and BKP hierarchies, discrete Darboux system and other discrete systems arise as the particular versions of the CSs which govern discrete and difference deformations of associative algebras. The difference CS has a simple geometrical meaning of vanishing 'discrete' Riemann curvature tensor with the structure constants C^l_{jk} playing the role of 'discrete' Christoffel symbols.

Such an interpretation of discrete equations allows us to better understand the algebraic backgrounds of the theory of discrete integrable systems and associated constructions in discrete geometry. We will show, for instance, that the DSKP equation or the Menelaus relation and the Hirota–Miwa bilinear equation for KP hierarchy are just the associativity conditions for the structure constants of certain algebras. The consistency around cube and multidimensional consistency discussed in discrete geometry also have a meaning of conditions of associativity for elements of algebras. On the other hand, the transfer of the old concept of gauge equivalency from the theory of integrable systems (see e.g. [25–28]) to deformation theory introduces the notion of gauge equivalence classes of deformations. In geometrical terms, this leads to the notion of the gauge equivalency between geometrical configurations. Menelaus and KP six-point configurations on the plane represent an important example of such situation.

Interpretation of discrete integrable systems as equations governing deformations of associative algebras provides us also with a method for construction of integrable discretizations of integrable PDEs. This problem has been intensively discussed for many years and several methods have been proposed (see e.g. [32–40]). In our approach, an integrable discretization is just the change of the DDA from the Heisenberg algebra to the algebra of shifts for an associative algebra in the given basis, i.e. for the same structure constants. Interinfluence of the theories of integrable systems and deformations of associative algebras are revealed to be rather fruitful.

The paper is organized as follows. In section 2, we briefly review some well-known integrable equations, namely the discrete Korteweg–de Vries (KdV) equation, discrete Schwarzian KdV (DSKdV) equation, DSKP equation, discrete Hirota–Miwa equation and their connection with the Backlund and Darboux transformations for continuous integrable equations. We discuss also the indications on the possible role of associative algebras in these constructions. In section 3, we present a general theory of deformations of structure constants for associative algebras. A closed left ideal generated by elements representing the multiplication table plays the central role in this construction. Deformations of structure constants are generated by the DDA and are governed by the corresponding CS. A subclass of deformations, the so-called integrable deformations, is discussed in section 4. The CS for such deformations has a geometrical meaning of vanishing discretized Riemann curvature tensor. Discrete deformations of the three-dimensional associative algebra and discrete Boussinesq equation are studied in section 5. Next, section 6 is devoted to the discrete and semi-discrete versions of the WDVV equation. Deformations generated by the three-dimensional Lie algebras and corresponding discrete mappings are considered in section 7. Discrete versions of the oriented associativity equation are discussed in section 8. Discrete deformations of algebras for which the product of only distinct elements of the basis is defined are studied in section 9. Deformations of the three-dimensional algebras of such type, Menelaus configurations and deformations, are considered in section 10. In section 11, the KP configurations, discrete KP deformations and their gauge equivalence to the Menelaus configurations and deformations are analysed. Section 12 is devoted to the multidimensional extensions of the Menelaus and KP configurations and deformations. Deformations governed by the discrete Darboux system and discrete B-type Kadomtsev–Petviashvili (BKP) Hirota–Miwa equation are considered in section 13.

2. Backlund–Darboux transformations, discrete integrable systems and algebras behind them

Associative algebras show up in various branches of the theory of integrable continuous and discrete systems. One of the simplest and, probably, algebraically the most transparent ways to establish a connection between the continuous and discrete integrable equations and to reveal a possible role of associative algebras in their constructions is provided by Backlund transformations (BTs) and Darboux transformations (DTs).

BTs and DTs are the discrete transformations (depending on parameters) which act on the variety of solutions of given integrable PDE (see e.g. [52, 53]). They commute and, as the consequence, one has the algebraic relations between several solutions of the original PDE which are usually referred to as the nonlinear superposition formulae (NSFs). Due to the commutativity, one can treat BTs and DTs as the shifts on the lattice and the corresponding NSF takes the form of the discrete equation on this lattice (see e.g. [34, 54]).

Probably, the first demonstration of the efficiency of this scheme is associated with the sine-Gordon equation

$$\varphi_{xy} = \sin\varphi \tag{4}$$

where $\varphi_x = \frac{\partial\varphi}{\partial x}$, etc. Introduced and well studied within the classical differential geometry of surfaces in R^3 (see e.g. [52, 53]) more than a century ago, this equation has been recognized as integrable by the inverse scattering transform (IST) method in 1973 [55, 56]. BT $\varphi \to \varphi_1 = B_{a_1}\varphi$ for the sine-Gordon equation is defined by the relations [52, 53, 57]

$$\frac{1}{2}(\varphi_1 - \varphi)_x = a_1 \sin\left(\frac{\varphi_1 + \varphi}{2}\right), \qquad \frac{1}{2}(\varphi_1 + \varphi)_y = \frac{1}{a_1}\sin\left(\frac{\varphi_1 - \varphi}{2}\right) \tag{5}$$

where a_1 is an arbitrary parameter. BTs (5) with a different commute $B_{a_1}B_{a_2} = B_{a_2}B_{a_1}$. This leads to the following NSF:

$$\tan\left(\frac{\varphi_{12}-\varphi}{4}\right) = \frac{a_2+a_1}{a_2-a_1}\tan\left(\frac{\varphi_2-\varphi_1}{4}\right) \tag{6}$$

where $\varphi_{12} = B_{a_1}B_{a_2}\varphi$. This pure algebraic relation between four solutions of equation (4) is a very useful one. It has allowed us to calculate all multisoliton solutions of the sine-Gordon equation many years before the discovery of the IST method [58]. Then, due to the commutativity of BTs one can treat B_{a_1} and B_{a_2} as the shifts T_1 and T_2 of the solution φ at fixed x and y, respectively: $T_1\varphi(x, y) = \varphi_1(x, y)$, $T_2\varphi(x, y) = \varphi_2(x, y)$, $T_1T_2\varphi(x, y) = \varphi_{12}(x, y)$. Enumerating the family of solutions of equation (4) obtained by all compositions of BTs by two integers n_1, n_2 such that $T_1\varphi(n_1, n_2) = \varphi(n_1 + 1, n_2)$, $T_2\varphi(n_1, n_2) = \varphi(n_1, n_2 + 1)$, etc, one rewrites the NSF (6) in the form of the discrete equation

$$\tan\left(\frac{T_1T_2\varphi-\varphi}{4}\right) = \frac{a_2+a_1}{a_2-a_1}\tan\left(\frac{T_2\varphi-T_1\varphi}{4}\right).$$

This equation represents the discretization of equation (4). Remarkably, it coincides (up to some trivial redefinitions) with the integrable discretization of the sine-Gordon equation proposed by Hirota in [33] within a completely different method.

There are many examples of such type. For the celebrated KdV equation

$$u_t + u_{xxx} - 6u_xu = 0,$$

the spatial part of BT $u \to u_1$ is given by [59]

$$(u_1+u)_x + (u_1-u)\sqrt{\alpha_1^2 - 2(u_1+u)} = 0 \tag{7}$$

where α_1 is a parameter. The corresponding NSF of term of the potential V defined by $u_x = V$ is [59]

$$V_{12} - V = \frac{(\alpha_1+\alpha_2)(V_2-V_1)}{\alpha_1-\alpha_2+V_2-V_1}. \tag{8}$$

Interpreted as the discrete equation, this NSF is exactly the discrete KdV equation

$$(\alpha_1 - \alpha_2 + (T_2 - T_1)V)\,(\alpha_1 + \alpha_2 - (T_1T_2 - 1)V) = \alpha_1^2 - \alpha_2^2$$

introduced in [36, 37] within the direct linearization approach. This discrete KdV equation is the generating equation for the whole KdV hierarchy.

In the same manner, one can construct the discrete modified KdV equation and discrete Schwarzian KdV equation [36–38]. The latter one is [38]

$$\frac{(\Phi - T_1\Phi)(T_1T_2 - T_2\Phi)}{(T_1\Phi - T_1T_2\Phi)(T_2\Phi-\Phi)} = \frac{q^2}{p^2} \tag{9}$$

where p and q are arbitrary real parameters.

Similar results are valid also for (1+1)-dimensional integrable systems of PDEs. For instance, for the AKNS hierarchy [60] the first member of which is the system

$$\mathrm{i}q_t + q_{xx} + 2q^2r = 0, \qquad \mathrm{i}r_t - r_{xx} - 2r^2q = 0, \tag{10}$$

one has two elementary BTs [61] (spatial parts)

$$B_\alpha^1 : \mathrm{i}q'_x - \tfrac{1}{2}q'^2r + 2\alpha q' + 2q = 0, \qquad \mathrm{i}r_x + \tfrac{1}{2}r^2q' - 2\alpha r - 2r' = 0, \tag{11}$$

$$B_\beta^2 : \mathrm{i}q_x - \tfrac{1}{2}q^2r' + 2\beta q + 2q' = 0, \qquad \mathrm{i}r'_x + \tfrac{1}{2}r'^2q - 2\beta r' - 2r = 0 \tag{12}$$

where α and β are arbitrary parameters. The NSF for these BTs consists of the following two equations [61]:

$$q_{12} = q + \frac{2(\alpha-\beta)}{\frac{r_2}{2}+\frac{2}{q_1}}, \qquad r_{12} = r - \frac{2(\alpha-\beta)}{\frac{q_1}{2}+\frac{2}{r_2}}. \tag{13}$$

Again it represents the discrete integrable AKNS system (10) and generates the whole AKNS hierarchy. Note that this NSF implies the discrete equation $\left(u=\frac{q}{2}, v=\frac{r}{2}\right)$

$$(T_1T_2u-u)\left(T_2v+\frac{1}{T_1u}\right)+(T_1T_2v-v)\left(T_1u+\frac{1}{T_2v}\right)=0. \tag{14}$$

Algebraic NSFs for (1+1)-dimensional integrable equations typically contain four solutions and consequently their discrete integrable versions usually are the four-point relations on a lattice.

For (2+1)-dimensional integrable PDEs, the situation is quite different. For instance, for the KP equation the analogue of the NSF (8) contains derivatives and more BT connected solutions are required in order to get a pure algebraic NSF. We will consider here the KP equation and hierarchy as an illustrative example. We will also use the technique based on DT to derive the NSF (see [48]).

We start with the standard linear problem

$$\psi_y = \psi_{xx} + u\psi \tag{15}$$

and adjoint linear problem

$$-\psi^*_y = \psi^*_{xx} + u\psi^* \tag{16}$$

for the KP hierarchy. Standard DTs are given by (see e.g. [8])

$$D_i : T_i\psi = \psi_x - \frac{\psi_{ix}}{\psi_i}\psi, \qquad T_iu = u + 2(\ln\psi_i)_{xx}, \qquad i=1,2,3, \tag{17}$$

where $\psi_i, i = 1, 2, 3$ are independent solutions of the problem (15) with the original potential u. The subsequent action of two DTs (17) with distinct ψ_i is of the form

$$T_kT_i\psi = T_k\psi_x - \frac{T_k\psi_{ix}}{T_k\psi_i}T_k\psi, \qquad i\neq k, \tag{18}$$

where

$$T_k\psi_i = \psi_{ix} - \frac{\psi_{kx}}{\psi_k}\psi_i, \qquad i\neq k. \tag{19}$$

DTs commute $T_iT_k = T_kT_i$. As the consequence, the elimination of ψ_i, ψ_{ix} and $T_k\psi_i$ from formulae (17), (18) gives the algebraic NSF [62]

$$T_1\left(\frac{(T_2-T_3)\psi}{\psi}\right)+T_2\left(\frac{(T_3-T_1)\psi}{\psi}\right)+T_3\left(\frac{(T_1-T_2)\psi}{\psi}\right)=0. \tag{20}$$

In a similar manner, one obtains the NSF for the wavefunction ψ^* of the adjoint problem (16):

$$\frac{(T_1-T_2)\psi^*}{T_1T_2\psi^*}+\frac{(T_2-T_3)\psi^*}{T_2T_3\psi^*}+\frac{(T_3-T_1)\psi^*}{T_3T_1\psi^*}=0. \tag{21}$$

For binary DT [8], ψ transforms as given in (17) and

$$T_i\psi^* = -\frac{\Phi_i}{\psi_i}, \qquad i=1,2,3, \tag{22}$$

where $\Phi_i \doteqdot \Phi(\psi_i, \psi^*)$ and the bilinear potential $\Phi \doteqdot \Phi(\psi, \psi^*)$ is defined by

$$\Phi_x = \psi\psi^*, \qquad \Phi_y = \psi^*\psi_x - \psi\psi^*_x. \tag{23}$$

For the potential Φ, the binary DTs are of the form

$$T_i\Phi = \Phi - \Phi_i\frac{\psi}{\psi_i}, \qquad T_kT_i\Phi = T_k\Phi - T_k\Phi_i\cdot\frac{T_k\psi}{T_k\psi_i}, \qquad i,k=1,2,3; \quad i\neq k, \tag{24}$$

where

$$T_k\Phi_i = \Phi_i - \Phi_k\frac{\psi_i}{\psi_k}. \tag{25}$$

Since $T_iT_k\Phi = T_kT_i\Phi$, the elimination of Φ_i, ψ_i and ψ_{ix} from the above formulae gives rise to the following NSF:

$$\frac{(T_1\Phi - T_1T_2\Phi)(T_2\Phi - T_2T_3\Phi)(T_3\Phi - T_3T_1\Phi)}{(T_1T_2\Phi - T_2\Phi)(T_2T_3\Phi - T_3\Phi)(T_3T_1\Phi - T_1\Phi)} = -1. \tag{26}$$

Due to the commutativity of DTs, one can interpret their action as the shifts on the lattice $T_1\Phi(n_1,n_2,n_3) = \Phi(n_1+1,n_2,n_3)$, etc and, consequently, the NSFs (20), (21) and (26) represent the discrete equations on the lattice. All of them contain six points of the lattice in contrast to the (1+1)-dimensional case. Under the constraint $\Phi_{23} = \Phi$, equation (26) is reduced to the discrete Schwarzian KdV equation (9) [48].

Discrete equations (20), (21), (26) are fundamental equations for the KP hierarchy. They generate the whole hierarchies. For instance, equation (26) is the generating equation for the Schwarzian KP hierarchy. Moreover, it has a beautiful geometrical meaning in connection with the classical Menelaus theorem [48].

Discrete equations (20), (21), (26) have been derived in a different way in the papers [62–65]. In particular, in [64, 65] it was shown that all the above equations arise as the compatibility conditions for the system

$$\Delta_i\Phi = \psi T_i\psi^*, \qquad i=1,2,3, \tag{27}$$

where $\Delta_i = T_i - 1$ and T_i are the Miwa shifts of the KP times, i.e. $T_i\Phi(t) = \Phi(t+[a_i]) = \Phi\big(t_1 + a_i, t_2 + \frac{1}{2}a_i^2, t_3 + \frac{1}{3}a_i^3, \ldots\big)$.

Equations (20), (21), (26) are closely connected with one more discrete equation associated with the KP hierarchy, namely with the famous bilinear Hirota–Miwa equation

$$T_1\tau\cdot T_2T_3\tau - T_2\tau\cdot T_3T_1\tau + T_3\tau\cdot T_2T_1\tau = 0 \tag{28}$$

for the τ-function. Solutions of equations (20), (21), (26) are ratios of τ-functions. The τ-function is sort of homogeneous coordinates for the lattice defined by these discrete equations.

One has discrete equations similar to equations (20), (21), (26), (28) for multicomponent KP hierarchy, two-dimensional Toda lattice (2DTL) hierarchy and other hierarchies [64, 65]. They also have a nice geometrical interpretation [49–51]. One of their common features is that all of them are six and more point relations on the lattice.

The connection between BTs and DTs and discrete integrable equations helps us also to clarify algebraic structures behind them. The IST linearizes not only the nonlinear PDEs integrable by the method but also the action of BTs and DTs [66–68]. For instance, the action of the BT (7) which adds one soliton to the solution of the KdV equation in term of the reflection coefficient $R(\lambda)$ (part of the inverse problem data) is a very simple one:

$$B_\alpha R(\lambda) = \frac{\lambda - \mathrm{i}\alpha}{\lambda + \mathrm{i}\alpha}R(\lambda). \tag{29}$$

Multiple action of BTs is then the multiplication by the rational function:

$$\prod_{k=1}^{n} B_{\alpha_k}\cdot R(\lambda) = \prod_{k=1}^{n}\frac{\lambda - \mathrm{i}\alpha_k}{\lambda + \mathrm{i}\alpha_k}R(\lambda). \tag{30}$$

An action of BTs for other integrable equations has a similar form (see e.g. [66–68]).

This property of BTs is inherited in the certain constructions of discrete integrable equations [35–38, 69, 70]. For example, the method proposed in [36–38] is based on the integral equation

$$\Psi(k) = \Psi_0(k) + \iint_D \Psi(l)\,\mathrm{d}\mu(l,l')G(k,l') \tag{31}$$

for the matrix-valued functions of the integers n_1, n_2, n_3. The shift in the variable n_i is generated by the multiplication of the measure $\mathrm{d}\mu(l,l')$ by a simple rational function:

$$T_i : \mathrm{d}\mu(l,l') \to \mathrm{d}\mu'(l,l') = \frac{l-p_i}{l'+p_i}\,\mathrm{d}\mu(l,l') \tag{32}$$

where p_i are parameters.

Within the $\overline{\partial}$-dressing method based on the nonlocal $\overline{\partial}$ problem [71] (see also [72])

$$\frac{\partial\Psi(\lambda)}{\partial\overline{\lambda}} = \iint \mathrm{d}\lambda'\Psi(\lambda')R(\lambda',\lambda), \tag{33}$$

an action of BT on the $\overline{\partial}$-data $R(\lambda',\lambda)$ is given by the formula [73, 74]

$$B_a R(\lambda',\lambda) = \frac{\lambda'-a}{\lambda-a}R(\lambda',\lambda). \tag{34}$$

Both the $\overline{\partial}$-dressing method and the direct linearization approach allow us to construct wide classes of discrete integrable equations. In both these methods, an action of multiple shifts on the corresponding data is represented by the multiplication by rational functions

$$\prod_{k=1}^{n} T_{\alpha_k}\cdot R(\lambda',\lambda) = \prod_{k=1}^{n}\frac{\lambda'-a_k}{\lambda-a_k}R(\lambda',\lambda). \tag{35}$$

The method of constructing discrete equations proposed in [35] uses, essentially, the same idea.

Family of rational functions provides us with several examples of associative algebras. One of them is the algebra of complex functions with simple poles in distinct points. In virtue of the identity

$$\frac{a_i}{\lambda-\lambda_i}\cdot\frac{a_k}{\lambda-\lambda_k} = A_i\frac{a_i}{\lambda-\lambda_i} + A_k\frac{a_k}{\lambda-\lambda_k}, \qquad i\neq k,$$

with $A_i = \frac{a_k}{\lambda_i-\lambda_k}$, $A_k = \frac{a_i}{\lambda_k-\lambda_i}$ where λ is a complex variable and $\lambda_i, \lambda_k, a_i, a_k$ are arbitrary parameters, the table of multiplication for the elements $\mathbf{P}_i$ of the basis of this algebra is of the form

$$\mathbf{P}_i\cdot\mathbf{P}_k = A_i\mathbf{P}_i + A_k\mathbf{P}_k, \qquad i\neq k, i,k = 1,2,\ldots,N. \tag{36}$$

Functions $\mathbf{P}_n = \frac{a_n}{(\lambda-\lambda_0)^n}$ with multiple poles at the same point form an infinite-dimensional associative algebra with the multiplication table $\mathbf{P}_n\cdot\mathbf{P}_m = \mathbf{P}_{n+m}$. A natural extension of this example to the polynomials $\mathbf{P}_j = \sum_{m=0}^{j}\frac{a_{jm}}{(\lambda-\lambda_0)^m}$ provides us with the infinite-dimensional associative algebra with the multiplication table

$$\mathbf{P}_j\cdot\mathbf{P}_k = \sum_{l=1}^{j+k} C_{jk}^{l}\mathbf{P}_l, \qquad j,k = 1,2,\ldots, \tag{37}$$

where C_{jk}^l are certain constants (structure constants). Under the additional polynomial constraint $\mathbf{P}_1^{N+1}+u_N\mathbf{P}_1^N+\cdots u_1\mathbf{P}_1 = 0$, the algebra (37) becomes the N-dimensional associative algebra.

Associative algebras of the type (36) and (37) show up in the study of many integrable systems both continuous and discrete. Within the methods mentioned above, the ring of rational functions and associative algebras did not play a significant role. The relations of the type (36), (37) have appeared only in certain intermediate calculations.

In a completely different context, associative algebras have been used for construction of integrable systems in papers [75–80]. For instance, in [80] they served basically to fix a domain of definition of dependent variables for ODEs.

In the rest of the paper we shall try to demonstrate that the simple associative algebras of the types (36), (37) are intimately connected with integrable systems. The latter arises as the equations describing deformations of the structure constants for such associative algebras.

3. Deformations of associative algebras

Here we will present basic elements of the approach to the deformations of structure constants for associative algebras proposed in [21–24] in a slightly modified form.

So, we consider a finite-dimensional noncommutative algebra A with (or without) unite element $\mathbf{P}_0$. We will restrict ourself to a class of algebras which possess a basis composed of pairwise commuting elements $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_N$. The table of multiplication

$$\mathbf{P}_j \cdot \mathbf{P}_k = \sum_{l=0}^{N} C_{jk}^{l} \mathbf{P}_l, \qquad j, k = 0, 1, \ldots, N, \tag{38}$$

defines the structure constants C_{jk}^l. The commutativity of the basis implies that $C_{jk}^l = C_{kj}^l$. In the presence of the unite element, one has $C_{j0}^l = \delta_j^l$ where δ_j^l is the Kroneker symbol.

Following Gerstenhaber's suggestion [3, 4], we will treat the structure constants C_{jk}^l in a given basis as the objects to deform and will denote the deformation parameters by $x^1, x^2, \ldots, x^M$. In the construction of deformations, we should first specify a 'deformed' version of the multiplication table (38) and then require that this realization is self-consistent and meaningful.

Thus, to define deformations we

(1) associate a set of elements $p_0, p_1, \ldots, p_N, x^1, x^2, \ldots, x^M$ with the elements of the basis $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_N$ and deformation parameters $x^1, x^2, \ldots, x^M$,

(2) consider the Lie algebra B of the dimension $N + M + 1$ with the basis elements $e_1, \ldots, e_{N+M+1}$ obeying the commutation relations

$$[e_\alpha, e_\beta] = \sum_{\gamma=1}^{N+M+1} C_{\alpha\beta\gamma} e_\gamma, \qquad \alpha, \beta = 1, 2, \ldots, N + M + 1, \tag{39}$$

(3) identify the elements $p_0, p_1, \ldots, p_N, x^1, x^2, \ldots, x^M$ with the elements $e_1, \ldots, e_{N+M+1}$ thus defining the deformation driving algebra (DDA). Different identifications define different DDAs. We will assume that the element p_0 is always a central element of the DDA. The commutativity of the basis in the algebra A implies the commutativity between p_j and in this paper we assume the same property for all x^k. So, we will consider the DDAs defined by the commutation relations of the type

$$\begin{gathered} [p_j, p_k] = 0, [x^j, x^k] = 0, [p_0, p_k] = 0, [p_0, x^k] = 0, \\ [p_j, x^k] = \sum_l \alpha_{jl}^k x^l + \sum_l \beta_j^{kl} p_l \end{gathered} \tag{40}$$

where α_{jl}^k and β_j^{kl} are some constants,

(4) consider the elements

$$f_{jk} = -p_j p_k + \sum_{l=0}^{N} C^l_{jk}(x) p_l, \qquad j,k = 0,1,\ldots,N, \tag{41}$$

of the universal enveloping algebra U(B) of the algebra DDA(B). These f_{jk} 'represent' the table (38) in U(B),

(5) require that the left ideal $J = \langle f_{jk} \rangle$ generated by these elements f_{jk} is closed

$$[J, J] \subset J \tag{42}$$

or, equivalently , that

$$[f_{jk}, f_{lm}] = \sum_{s,t=0}^{N} K^{st}_{jklm} \cdot f_{st}, \qquad j,k,l,m = 0,1,\ldots,N, \tag{43}$$

where K^{st}_{jklm} are some elements of U(B).

Definition. *The structure constants $C^l_{jk}(x)$ are said to define deformations of the algebra A generated by the given DDA if the left ideal $J = \langle f_{jk} \rangle$ is closed.*

To justify this definition, we observe that the simplest possible realization of the multiplication table (38) in U(B) given by the equations $f_{jk} = 0$ is too restrictive. Indeed, the commutativity of p_j implies in this case that $\left[p_t, C^l_{jk}(x)\right] = 0$ and, hence, no deformations are allowed. So, one should look for a weaker realization of the multiplication table which is self-consistent. The condition that the set of f_{jk} form a closed 'algebra' (43) is a natural candidate.

The condition (43) implies certain constraints on the structure constants. The use of relations (40) provides us with the following identities:

$$[f_{jk}, f_{lm}] = \sum_{s,t=0}^{N} K^{st}_{jklm}(x,p) \cdot f_{st} + \sum_{t=0}^{N} N^t_{jklm}(x) \cdot p_t, \qquad j,k,l,m = 0,1,\ldots,N, \tag{44}$$

and

$$(p_j p_k)p_l - p_l(p_k p_l) = \sum_{s,t=0}^{N} L^{st}_{klj}(x,p) \cdot f_{st} + \sum_{t=0}^{N} \Omega^t_{klj}(x) \cdot p_t, \qquad j,k,l = 0,1,\ldots,N, \tag{45}$$

where $K^{st}_{jklm}(x,p)$, $N^s_{jklm}(x)$, $L^{st}_{klj}(x,p)$, $\Omega^t_{klj}(x)$ are certain elements of U(B). As an obvious consequence of the identity (44) one has

Proposition 1. *Structure constants $C^l_{jk}(x)$ define deformations generated by the DDA if they obey the system of equations*

$$N^t_{jklm}(x) = 0, \qquad j,k,l,m,s = 0,1,\ldots,N. \tag{46}$$

Concrete form of $K^{st}_{jklm}(x,p)$, $N^t_{jklm}(x)$, $L^{st}_{klj}(x,p)$, $\Omega^t_{klj}(x)$ and equations (46) is defined by the DDA (40). In this paper, we will consider as DDAs some three-dimensional Lie algebras and algebras defined by the following commutation relations:

$$[p_j, p_k] = 0, \qquad [x^j, x^k] = 0, \qquad [p_j, x^k] = \delta^k_j p_j, \qquad j,k = 1,2,\ldots,N, \tag{47}$$

and

$$[p_j, p_k] = 0, \qquad [x^j, x^k] = 0, \qquad [p_j, x^k] = \delta^k_j (p_0 + \varepsilon_j p_j), \qquad j,k = 1,2,\ldots,N, \tag{48}$$

where ε_j are arbitrary parameters. In what follows, we will put the central element equal to the unite element $\widehat{I}$. The algebra of shifts $p_j = T_j$ where $T_j x^k = x^k + \delta_j^k$, $T_j\varphi(x^1,\dots,x^N) = \varphi(x^1,\dots,x^j+1,\dots,x^N)$ is a realization of the algebra (47). A realization of the algebra (48) is given by the algebra of differences $p_j = \Delta_j = \frac{T_j-\widehat{I}}{\varepsilon_j}$ where $T_j x^k = x^k + \varepsilon_j\delta_j^k$. The family of algebras (48) contains the Heisenberg algebra as the limit when all $\varepsilon_j \to 0$. In this case $\Delta_j \to \frac{\partial}{\partial x^j}$. At $\varepsilon_j = 1$ $(j = 1,\dots,N)$ one has the algebra connected with the algebra (47) by the change of the basis $\mathbf{P}_j \longleftrightarrow \mathbf{P}_j + \mathbf{P}_0$ in the algebra A.

In order to calculate explicitly the rhs in the identities (44) and (45), one needs to know the commutator $\left[p_t, C_{jk}^l(x)\right]$. For the algebra (47) (i.e. the DDA (47)) and an element $\varphi(x^1,\dots,x^N) \subset U(\mathrm{DDA}(47))$, one has

$$[p_j,\varphi(x)] = \Delta_j\varphi(x)\cdot p_j, \qquad j = 1,\dots,N, \tag{49}$$

where $\Delta_j = T_j - 1$ and T_j is the shift operator $T_j x^k = x^k + \delta_j^k$. For the DDA (48), the analogous identity is

$$[p_j,\varphi(x)] = \Delta_j\varphi(x)\cdot(\widehat{I} + \varepsilon_j p_j), \qquad j = 1,\dots,N, \tag{50}$$

where $\Delta_j = \frac{T_j-\widehat{I}}{\varepsilon_j}$ and $T_j x^k = x^k + \varepsilon_j\delta_j^k$.

Using (49), for the DDA (47) one gets

$$K_{jklm}^{st} = \frac{1}{2}\Bigg(\delta_j^s\delta_k^t\sum_n (T_jT_kC_{lm}^n)p_n + (T_jT_kC_{lm}^s)(T_sC_{jk}^t) + \delta_l^s\delta_m^t\sum_n C_{jk}^n p_n - (j,l)(k,m) + (s,t)\Bigg)$$

where the variables in the parentheses (j,k) denote the previous terms with the exchange of indices indicated in the parentheses:

$$N_{jklm}^t = \sum_{n,s}\left(T_lT_mC_{jk}^s\right)\left(T_sC_{mn}^n\right)C_{ns}^t - \sum_{n,s}\left(T_jT_kC_{lm}^s\right)\left(T_sC_{jk}^n\right)C_{ns}^t \tag{51}$$

and

$$L_{klj}^{st} = \frac{1}{2}\left(\delta_k^s\delta_l^t p_j - \delta_j^s\delta_k^t p_l + \delta_j^s\left(T_jC_{kl}^t\right) - \delta_l^s\left(T_lC_{jk}^t\right) + (s,t)\right), \tag{52}$$

$$\Omega_{klj}^t = \sum_s\left(C_{ls}^t\left(T_lC_{jk}^s\right) - C_{sj}^t\left(T_jC_{kl}^s\right)\right). \tag{53}$$

For the DDA (48), the use of (50) gives

$$\begin{aligned} K_{jklm}^{st} &= \frac{1}{2}\Bigg(\delta_j^s\delta_k^t\sum_n\left(T_jT_kC_{lm}^n\right)p_n + \left(T_jT_kC_{lm}^s\right)\left(T_sC_{jk}^t\right) \\ &\quad + \delta_j^t\left(T_j\Delta_kC_{lm}^s\right) + \delta_k^t\left(T_k\Delta_jC_{lm}^s\right) - (j,l)(k,m) + (s,t)\Bigg), \\ N_{jklm}^t &= -\Delta_j\Delta_kC_{lm}^t - \sum_s\Bigg(C_{js}^tT_j\Delta_kC_{lm}^s + C_{ks}^tT_k\Delta_jC_{lm}^s \\ &\quad + \left(T_jT_kC_{lm}^s\right)\left(\Delta_sC_{jk}^t\right) + \sum_n\left(T_jT_kC_{lm}^s\right)\left(T_sC_{jk}^n\right)C_{ns}^t\Bigg) - (j,l)(k,m) \end{aligned} \tag{54}$$

and

$$L_{klj}^{st} = \frac{1}{2}\left(\delta_k^s\delta_l^t p_j - \delta_j^s\delta_k^t p_l + \delta_j^s\left(\Delta_jC_{kl}^t\right) - \delta_l^s\left(\Lambda_lC_{jk}^t\right) + (s,t)\right), \tag{55}$$

$$\Omega^t_{klj} = \Delta_l C^t_{jk} - \Delta_j C^t_{lk} + \sum_s \left(C^t_{ls}\left(T_l C^s_{jk}\right) - C^t_{sj}\left(T_j C^s_{kl}\right)\right). \tag{56}$$

Thus, deformations generated by DDAs (47) and (48) are governed by equations (46) with N^t_{jklm} given by (52) and (53), respectively.

4. Class of integrable deformations

The lhs of equation (46) has a special structure in both cases under consideration. Indeed, one can show that for the DDA (47)

$$N^t_{jklm} = \frac{1}{2}\sum_s \Big(C^t_{ls}\left(T_l \Omega^s_{kmj}\right) - C^t_{js}\left(T_j \Omega^s_{mkl}\right) - \left(T_j T_l C^s_{km}\right)\Omega^t_{sjl} + \left(T_j T_k C^s_{lm}\right)\Omega^t_{ksj} + \left(T_m T_l C^s_{jk}\right)\Omega^t_{msl} + (j,k)(l,m)\Big) \tag{57}$$

while for the DDA (48) one has

$$N^t_{jklm} = \Delta_m \Omega^t_{jlk} - \Delta_k \Omega^t_{ljm} + \frac{1}{2}\sum_s \Big(C^t_{ks}\left(T_k \Omega^s_{lmj}\right) + C^t_{ms}\left(T_m \Omega^s_{jlk}\right) + \left(T_j T_k C^s_{lm}\right)\Omega^t_{jks} + \left(T_l T_m C^s_{jk}\right)\Omega^t_{lsm} + \left(T_k T_m C^s_{jl}\right)\Omega^t_{smk} + (j,k)(l,m)\Big). \tag{58}$$

The rhs of these formulae vanish if the structure constants obey the equation $\Omega^t_{klj} = 0$. Thus, one has

Proposition 2. *The equations*

$$\sum_s \left(C^t_{ls}\left(T_l C^s_{jk}\right) - C^t_{sj}\left(T_j C^s_{kl}\right)\right) = 0 \tag{59}$$

and

$$\Delta_l C^t_{jk} - \Delta_j C^t_{lk} + \sum_s \left(C^t_{ls}\left(T_l C^s_{jk}\right) - C^t_{sj}\left(T_j C^s_{kl}\right)\right) = 0 \tag{60}$$

govern the subclasses of deformations generated by the DDA (47) and DDA (48), respectively.

In the rest of the paper, we will study only these classes of deformations. We will refer to the systems of equations (59), (60) as the central systems (CSs). There are at least two reasons to consider these subclasses of deformations. The first is that they are equivalent to the compatibility conditions of the linear systems

$$f_{jk}\Psi = 0, \qquad j, k, = 1, \ldots, N, \tag{61}$$

where Ψ is a common right divisor of zero for all f_{jk}. Recall that nonzero elements a and b of an algebra are called left and right divisors of zero if $ab = 0$ (see e.g. [81]). Within the interpretation of p_j and x^k as operators acting in a linear space H equations (61) become the following linear problems for integrable systems:

$$\left(-p_j p_k + \sum_l C^l_{jk}(x) p_l\right)|\Psi\rangle = 0, \qquad j, k = 1, \ldots, N, \tag{62}$$

where $|\Psi\rangle \subset H$. So, one can refer to such deformations as integrable one.

The second reason to study equations (59), (60) is that they have a nice geometrical meaning. We begin this study with rewriting these equations in a compact form. Introducing the operators T^C_j and ∇_j acting as

$$T^C_j \Phi^n_k = \sum_s C^n_{js} T_j \Phi^s_k \tag{63}$$

and

$$\nabla_j \Phi_k^n = \Delta_j \Phi_k^n + \sum_s C_{js}^n T_j \Phi_k^s, \tag{64}$$

one gets the following form of the formulae (51) and (54):

$$N_{jklm}^t = \sum_n \left((T_l T_m C_{jk}^n) T_n^C C_{lm}^t - (T_j T_k C_{lm}^n) T_n^C C_{jk}^t \right) \tag{65}$$

and

$$N_{jklm}^t = \nabla_l \nabla_m C_{jk}^t - \nabla_j \nabla_k C_{lm}^t + \sum_n \left(T_l T_m C_{jk}^n\right)\left(\nabla_l C_{nm}^t - \nabla_m C_{nl}^t\right) \\ - \sum_n \left(T_j T_k C_{lm}^n\right)\left(\nabla_j C_{nk}^t - \nabla_k C_{nj}^t\right), \tag{66}$$

respectively. For Ω_{klj}^t (53), (56), one gets

$$\Omega_{klj}^t = T_l^C C_{jk}^t - T_j^C C_{lk}^t \tag{67}$$

and

$$\Omega_{klj}^t = \nabla_l C_{jk}^t - \nabla_j C_{lk}^t. \tag{68}$$

Then, introducing the matrices C_j and Ω_{lj} such that $(C_j)_k^l = C_{jk}^l$ and $(\Omega_{lj})_k^t = \Omega_{klj}^t$, one rewrites equations (59) and (60) in the matrix form

$$\Omega_{lj} = C_l T_l C_j - C_j T_j C_l = T_l^C C_j - T_j^C C_l = 0 \tag{69}$$

and

$$\Omega_{lj} = \Delta_l C_j - \Delta_j C_l + C_l T_l C_j - C_j T_j C_l = \nabla_l C_j - \nabla_j C_l = 0. \tag{70}$$

Note that equation (70) is equivalent to the equation

$$(1 + \varepsilon_l C_l) T_l (1 + \varepsilon_j C_j) - (1 + \varepsilon_j C_j) T_j (1 + \varepsilon_l C_l) = 0, \tag{71}$$

which is of the form (69) for the matrix $\widetilde{C}_j = 1 + \varepsilon_j C_j$. One observes this similarity also for the operators T_j^C and ∇_j which in the matrix notations act as follows:

$$T_j^C = C_j T_j, \nabla_j = \Delta_j + C_j T_j = \frac{1}{\varepsilon_j}((1 + \varepsilon_j C_j) T_j - 1). \tag{72}$$

For constant structure constants CSs (69), (60) are reduced to the associativity condition (3), i.e. $[C_l, C_j] = 0$. In general, for deformed $C_{jk}^l(x)$ this condition is not satisfied. The defect of associativity or quantum anomaly for deformations [21] is defined as the matrix $\alpha_{lj} = \Omega_{lj} - [C_l, C_j]$. For deformations generated by the DDA (47), it is equal to

$$\alpha_{lj} = C_l \Delta_l C_j - C_j \Delta_j C_l \tag{73}$$

while for the DDA (48)

$$\alpha_{lj} = (1 + \varepsilon_l C_l) \Delta_l C_j - (1 + \varepsilon_j C_j) \Delta_j C_l. \tag{74}$$

To clarify the geometrical content of equation (70), we note that in the case of all $\varepsilon_j = 0$ it is reduced to that of quantum deformations [21], i.e. to the system

$$\Omega_{klj}^t = \frac{\partial C_{kj}^t}{\partial x^l} - \frac{\partial C_{kl}^t}{\partial x^j} + \sum_m \left(C_{jk}^m C_{lm}^t - C_{lk}^m C_{jm}^t\right) = 0, \qquad j, k, l, t = 1, \ldots, N. \tag{75}$$

This equation has a geometrical meaning of vanishing Riemann curvature tensor $\left(R^{\text{class}}_{lj}\right)^t_k \doteqdot R^t_{klj} = \Omega^t_{klj}$ with the Christoffel symbols identified with the structure constants C^l_{jk} [21]. The operator ∇_j becomes a covariant derivative $\nabla_j = \frac{\partial}{\partial x^j} + C_j$ and one has (see e.g. [82])

$$R^{\text{class}}_{jk} = [\nabla_j, \nabla_k]. \tag{76}$$

In particular, the equation $R^{\text{class}}_{lj} = 0$ is equivalent to the compatibility condition for the linear problems

$$\nabla_j \Psi = 0, \qquad j = 1, \ldots, N. \tag{77}$$

For a general DDA (48), one observes that

$$[\nabla_j, \nabla_k] = \Omega_{jk} T_j T_k. \tag{78}$$

By analogy with (76), equation (78) can be understood as the definition of the discrete version R^d_{jk} of the curvature tensor R^{class}_{jk}:

$$[\nabla_j, \nabla_k] = R^d_{jk} T_j T_k. \tag{79}$$

Thus, $R^d_{jk} = \Omega_{jk}$ or in components

$$R^{dt}_{klj} = \Omega^t_{klj} = \Delta_l C^t_{jk} - \Delta_j C^t_{lk} + \sum_s \left(C^t_{ls}\left(T_l C^s_{jk}\right) - C^t_{sj}\left(T_j C^s_{kl}\right)\right). \tag{80}$$

Obviously, $\lim_{\varepsilon_j \to 0} R^{dt}_{klj} = R^{(class)t}_{klj}$.

Similar to the continuous case, the CS (70) is equivalent to the equation $[\nabla_l, \nabla_j] = 0$ and to the compatibility condition for the linear problems

$$\nabla_j \Psi = (\Delta_j + C_j T_j)\Psi = 0, \qquad j = 1, \ldots, N. \tag{81}$$

Amazingly, the 'tensor' (80) and the operator ∇_j (64) essentially coincide with the discrete Riemann curvature tensor and covariant derivative introduced earlier within various discretizations of Riemann geometry (see e.g. [83–85]). Thus, the CS which governs the deformations of the structure constants $C^l_{jk}(x)$ has the same geometrical meaning of vanishing Riemann curvature tensor both in continuous and difference cases.

Finally, we note that the CSs (59), (60) or (69), (70) are underdetermined systems of equations. Similar to the integrable PDEs (see e.g. [25–28]), it is connected with the gauge freedom. It is not difficult to see that, for instance, equation (69) is invariant under transformations

$$C_j \to \widetilde{C}_j = G C_j T_j G^{-1}, \tag{82}$$

where $G(x)$ is a diagonal matrix with the diagonal elements $G_k(x) \subset U(DDA)$ generated only by the elements $x^1, \ldots, x^N$, since under this transformation

$$\widetilde{\Omega}_{lj} \doteqdot \widetilde{C}_l T_l \widetilde{C}_j - \widetilde{C}_j T_j \widetilde{C}_l = G \Omega_{lj} T_l T_j G^{-1}. \tag{83}$$

The relation $C^l_{jk} = C^l_{kj}$ implies that $G_k = T_k g(x)$ where $g(x)$ is an arbitrary element of $U(DDA)$ generated by $x^1, \ldots, x^N$. So, the CS (69) is invariant under the transformations

$$C^l_{jk} \to \widetilde{C}^l_{jk} = T_l g \cdot (T_j T_k g^{-1}) C^l_{jk} \tag{84}$$

and for the elements f_{jk} one has

$$f_{jk} \to \widetilde{f}_{jk} \doteqdot -p_j p_k + \sum_l \widetilde{C}^l_{jk}(x) p_l = (T_j T_k g^{-1}) \cdot f_{jk} \cdot g. \tag{85}$$

Analogously, the CS (70) is invariant under the transformations

$$C_j \to \widetilde{C}_j = G \Delta_j G^{-1} + G C_j T_j G^{-1}. \tag{86}$$

In the continuous case $\left(\varepsilon_j \to 0, T_j \to 1, \Delta_j \to \frac{\partial}{\partial x^j}\right)$, the transformations (86) are well known in the theory of integrable equations as the gauge transformations. Transformations (84), (86) are their discrete and difference counterparts (see also [83–85]). The invariance of the CSs under these transformations means that deformations governed by them form the classes of gauge equivalent deformations. Note that the associativity conditions (3) themselves are not invariant under gauge transformations (84).

5. Discrete deformations of the three-dimensional algebra and Boussinesq equation

A simplest nontrivial example of the proposed scheme corresponds to the three-dimensional algebra with the unite element and the basis $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$. The table of multiplication is given by the trivial part $\mathbf{P}_0\mathbf{P}_j = \mathbf{P}_j$, $j = 0, 1, 2$, and by

$$\begin{aligned} \mathbf{P}_1^2 &= A\mathbf{P}_0 + B\mathbf{P}_1 + C\mathbf{P}_2, \\ \mathbf{P}_1\mathbf{P}_2 &= D\mathbf{P}_0 + E\mathbf{P}_1 + G\mathbf{P}_2, \\ \mathbf{P}_2^2 &= L\mathbf{P}_0 + M\mathbf{P}_1 + N\mathbf{P}_2 \end{aligned} \tag{87}$$

where the structure constants $A, B, \ldots, N$ depend only on the deformation parameters x^1, x^2. It is convenient also to arrange the structure constants $A, B, \ldots, N$ into the matrices C_1, C_2 defined as above by $(C_j)^l_k = C^l_{jk}$. One has

$$C_1 = \begin{pmatrix} 0 & A & D \\ 1 & B & E \\ 0 & C & G \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & D & L \\ 0 & E & M \\ 1 & G & N \end{pmatrix}. \tag{88}$$

In terms of these matrices, the associativity conditions (2) are written as

$$C_1C_2 = C_2C_1 \tag{89}$$

and the CSs (69) and (70) are

$$C_1T_1C_2 = C_2T_2C_1 \tag{90}$$

and

$$\Delta_1C_2 - \Delta_2C_1 + C_1T_1C_2 - C_2T_2C_1 = 0, \tag{91}$$

respectively.

Let us consider first the CS (91). In terms of $A, B, \ldots,$ it is the system

$$\begin{aligned} &\Delta_1D - \Delta_2A + AE_1 + DG_1 - DB_2 - LC_2 = 0, \\ &\Delta_1E - \Delta_2B + D_1 + BE_1 + EG_1 - EB_2 - MC_2 = 0, \\ &\Delta_1G - \Delta_2C + CE_1 + GG_1 - A_2 - GB_2 - NC_2 = 0, \\ &\Delta_1L - \Delta_2D + AM_1 + DN_1 - DE_2 - LG_2 = 0, \\ &\Delta_1M - \Delta_2E + L_1 + BM_1 + EN_1 - EE_2 - MG_2 = 0, \\ &\Delta_1N - \Delta_2G + CM_1 + GN_1 - D_2 - GE_2 - NG_2 = 0 \end{aligned} \tag{92}$$

where $A_j \doteqdot T_jA$, etc. As in the continuous case [21], we consider the constraint $\mathrm{B} = 0, \mathrm{C} = 1, \mathrm{G} = 0$. The first three equations (92) give

$$L = \Delta_1D - \Delta_2A + AE_1, \qquad M = \Delta_1E + D_1, \qquad N = E_1 - A_2 \tag{93}$$

and the rest of the system (92) takes the form

$$\begin{aligned}
&\big(\Delta_1^2-\Delta_2+E_{11}-E_2-A_{12}\big)E+2\Delta_1 D_1-\Delta_2 A_1+A_1E_{11}=0,\\
&\big(\Delta_1^2-\Delta_2+E_{11}-E_2-A_{12}\big)D+(-\Delta_1\Delta_2+\Delta_1E_1+D_{11})A+\Delta_1(AE_1)=0,\\
&\Delta_1(2E_1-A_2)+D_{11}-D_2=0.
\end{aligned}\tag{94}$$

In the continuous case $\varepsilon_1=\varepsilon_2=0$, i.e. for the Heisenberg DDA the CS (94) becomes $(A_{x_j}=\frac{\partial A}{\partial x^j}$, etc)

$$\begin{aligned}
&E_{x_1x_1}-E_{x_2}+2D_{x_1}-A_{x_2}=0,\\
&D_{x_1x_1}-D_{x_2}-A_{x_1x_2}+AE_{x_1}+(AE)_{x_1}=0,\qquad 2E-A=0
\end{aligned}\tag{95}$$

where all integration constants have been chosen to be equal to zero. Hence, one has the system

$$\tfrac{1}{2}A_{x_1x_1}-\tfrac{3}{2}A_{x_2}+2D_{x_1}=0,\qquad D_{x_1x_1}-D_{x_2}-A_{x_1x_2}+\tfrac{3}{4}(A^2)_{x_1}=0.\tag{96}$$

Eliminating D, one gets the Boussinesq (BSQ) equation

$$A_{x_2x_2}+\tfrac{1}{3}A_{x_1x_1x_1x_1}-(A^2)_{x_1x_1}=0.\tag{97}$$

The BSQ equation (97) defines quantum deformations of the structure constants in (87) with $B=0,C=1,G=0,L=D_{x_1}-A_{x_2}+\frac{1}{2}A^2,M=\frac{1}{2}A_{x_1}+D,N=-\frac{1}{2}A$ and A,D defined by (96).

In the pure discrete case $\varepsilon_1=\varepsilon_2=1$, the CS (94) represents the discrete version of the BSQ system (96) which is equivalent to that proposed in [37]. It defines the discrete deformations of the same structure constants.

There are also the mixed cases. The first is $\varepsilon_1=0,\varepsilon_2=1(T_1=1,\Delta_1=\frac{\partial}{\partial x^1},\Delta_2=T_2-1)$. The CS (94) is

$$\begin{aligned}
&E_{x_1x_1}+2D_{x_1}-(1+E)\Delta_2(E+A)=0,\\
&D_{x_1x_1}-D_{x_2}+AE_{x_1}+(AE)_{x_1}-\Delta_2A_{x_1}-D\Delta_2(E+A)=0,\\
&(2E-A_2)_{x_1}-\Delta_2D_2=0.
\end{aligned}\tag{98}$$

The second case corresponds to $\varepsilon_1=1,\varepsilon_2=0\big(\Delta_1=T_1-1,T_2=1,\Delta_2=\frac{\partial}{\partial x^2}\big)$ and the CS takes the form

$$\begin{aligned}
&\Delta_1^2E+(E_{11}-E-A_1)E+A_1E_{11}+2\Delta_1D_1-(E+A_1)_{x_2}=0,\\
&\Delta_1^2D+(E_{11}-E-A_1)D+A\Delta_1E_1+\Delta_1(AE_1)+D_{11}A-\Delta_1A_{x_2}-D_{x_2}=0,\\
&\Delta_1(2E_1-A)+D_{11}-D=0.
\end{aligned}\tag{99}$$

Equations (62) provide us with the linear problems for the CS (94). They are

$$\big(\Delta_2-\Delta_1^2+A\big)|\Psi\rangle=0,\qquad(\Delta_1\Delta_2-E\Delta_1-D)|\Psi\rangle=0\tag{100}$$

or equivalently

$$\big(\Delta_2-\Delta_1^2+A\big)|\Psi\rangle=0,\qquad\big(\Delta_1^3-(E+A)\Delta_1-(D+\Delta_1A)\big)|\Psi\rangle=0.\tag{101}$$

One can check that the equation $f_{22}|\Psi\rangle=0$ is a consequence of (100).

In the quantum case $\big(\Delta_j=\frac{\partial}{\partial x^j}\big)$, the system (101) is the well-known linear system for the continuous BSQ equation [86]. We note that in this case the third equation $f_{22}|\Psi\rangle=0$, i.e. $\big(\Delta_2^2-L-D\Delta_1+\frac{1}{2}A\big)|\Psi\rangle=0$, is the consequence of two equations (101).

In the pure discrete case, equations (101) represent the linear problems for the discrete BSQ equation. The first mixed case considered above can be treated as the BSQ equation with the discrete time. The second case instead represents a continuous isospectral flow for the third-order difference problem, i.e. a sort of the difference BSQ equation with continuous time.

We would like to emphasize that equations (96), (99) and (92) describe different classes of deformations of the same set of the structure constants $A, B, \ldots, N$ defined by the table (87).

Now let us consider the CS (90). It has the form (92) where the terms with the differences $\Delta_j A$, etc should be dropped out. In the BSQ gauge $B = 0$, $C = 1$, $G = 0$ it becomes

$$\begin{aligned}&(E_{11} - E_2 - A_{12})E + A_1 E_{11} = 0,\\ &(E_{11} - E_2 - A_{12})D + A_1 D_{11} = 0, \qquad D_{11} - D_2 = 0.\end{aligned} \tag{102}$$

Note that this system implies that $A_1 E_{11} D - AED_{11} = 0$. With the choice D = 1, the system (102) takes the form

$$AE = (AE_1)_1, \qquad A_{12} - A = E_{11} - E_2. \tag{103}$$

Introducing the function U such that $A = U_{11} - U_2$, $E = U_{12} - U$, one rewrites the system (103) as the single equation

$$(U_{12} - U)(U_{11} - U_2) = ((U_{112} - U_1)(U_{11} - U_2))_1.$$

This discrete equation governs deformations of the BSQ structure constants generated by the DDA (47).

6. WDVV, discrete and continuous-discrete WDVV equations

Another interesting reduction of the CS (92) corresponds to the constraint $C = 1$, $G = 0$, $N = 0$. In this case the CS is

$$\begin{aligned}&\Delta_1 D - \Delta_2 A + AE_1 - DB_2 - L = 0,\\ &\Delta_1 E - \Delta_2 B + D_1 + BE_1 - EB_2 - M = 0,\\ &\Delta_1 L - \Delta_2 D + AM_1 - DE_2 = 0,\\ &\Delta_1 M - \Delta_2 E + L_1 + BM_1 - EE_2 = 0,\\ &E_1 - A_2 = 0, M_1 - D_2 = 0.\end{aligned} \tag{104}$$

Third and sixth equations (104) imply the existence of the functions U and V such that

$$A = U_1, \qquad E = U_2, \qquad D = V_1, \qquad M = V_2. \tag{105}$$

Substituting the expressions for L and M given by the first two equations (104), i.e.

$$L = \Delta_1 V_1 - \Delta_2 U_1 + U_1 U_{12} - V_1 B_2, \qquad M = \Delta_1 U_2 - \Delta_2 B + V_{11} + BU_{12} - U_2 B_2$$

into the rest of the system , one gets the following three equations:

$$\begin{aligned}&\Delta_1 U_1 - \Delta_2 B + V_{11} + BU_{12} - U_2 B_2 - V_2 = 0,\\ &\left(\Delta_1^2 - \Delta_2\right)V_1 - \Delta_1\Delta_2 U_1 + \Delta_1(U_1 U_{12}) - \Delta_1(V_1 B_2) - V_1 U_{22} + U_1 V_{12} = 0,\\ &\Delta_1(V_2 + V_{11}) - \Delta_2(U_2 + U_{11}) + U_{11}U_{112} - U_2 U_{22} - V_{11}B_{12} + BV_{12} = 0.\end{aligned} \tag{106}$$

Solution of this system together with formulas (105) defines deformations of the structure constants $A, B, \ldots$ generated by the DDA (48).

In the pure continuous case ($\varepsilon_1 = \varepsilon_2 = 0$), the system (106) takes the form

$$\begin{aligned}&U_{x_1} - B_{x_2} = 0, \qquad V_{x_1} - U_{x_2} = 0,\\ &\left(V_{x_1} + U_{x_2} + U^2 - VB\right)_{x_1} - V_{x_2} = 0.\end{aligned} \tag{107}$$

This system of three conservation laws implies the existence of the function F such that

$$U = F_{x_1x_1x_2}, \qquad V = F_{x_1x_2x_2}, \quad B = F_{x_1x_1x_1}. \tag{108}$$

In terms of F the system (107) is

$$F_{x_2x_2x_2} - (F_{x_1x_1x_2})^2 + F_{x_1x_1x_1}F_{x_1x_2x_2} = 0. \tag{109}$$

It is the WDVV equation (1) with the metric $g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ [10, 11]. It is a well-known fact the WDVV equation describes deformations of the three-dimensional algebra (87) under the reduction $C = 1$, $G = 0$, $N = 0$ [3–10]. So, the system (106) represents the generalization of the WDVV equation to the case of deformations of the same algebra generated by the DDA (48).

In the pure discrete case $\varepsilon_1 = \varepsilon_2 = 1$, the system (106) gives us a pure discrete version of the WDVV equation. In the first mixed case $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, we have the system

$$\begin{aligned} &U_{x_1} - \Delta_2(B + V) - U_2\Delta_2 B = 0, \\ &\big(V_{x_1} + UU_2 - VB_2\big)_{x_1} - \Delta_2\big(V + U_{x_1}\big) - VU_{22} + UV_2 = 0, \\ &(V + V_2)_{x_1} - \Delta_2(U + U_2 + UU_2) - VB_2 + BV_2 = 0 \end{aligned} \tag{110}$$

while in the case $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ one gets

$$\begin{aligned} &\Delta_1 U_1 + B\Delta_1 U + V_{11} - V - B_{x_2} = 0, \\ &\Delta_1(\Delta_1 V_1 - U_{1x_2} + U_1^2 - V_1 B) - V_1 U + U_1 V_1 - V_{1x_2} = 0, \\ &\Delta_1(V + V_{11}) + U_{11}^2 - U^2 - V_{11}B_1 + BV_1 - (U + U_{11})_{x_2} = 0. \end{aligned} \tag{111}$$

The equations $f_{jk}|\Psi\rangle = 0$ for the system (104) in the coordinate representation $p_j = \Delta_j$ have the form

$$\begin{aligned} &\big(\Delta_2 - \Delta_1^2 + B\Delta_1 + A\big)|\Psi\rangle = 0, \qquad (\Delta_1\Delta_2 - E\Delta_1 - D)|\Psi\rangle = 0, \\ &\big(\Delta_2^2 - M\Delta_1 - L\big)|\Psi\rangle = 0. \end{aligned} \tag{112}$$

The system (112) in its turn is equivalent to the following:

$$\begin{aligned} &\big(\Delta_1^3 - B\Delta_1^2 - (\Delta_1 B + A_1 + E)\Delta_1 - (\Delta_1 A + D)\big)|\Psi\rangle = 0, \\ &\big(\Delta_2 - \Delta_1^2 + B\Delta_1 + A\big)|\Psi\rangle = 0, \qquad \big(\Delta_2^2 - M\Delta_1 - L\big)|\Psi\rangle = 0. \end{aligned} \tag{113}$$

The compatibility condition for the above linear problems is equivalent to the 'discretized' WDVV systems (104) or (106). In particular, in the pure continuous case the problems (112) in the coordinate representation coincide with the well-known one [12, 13] , namely

$$\begin{aligned} \Psi_{x_1x_1} &= F_{x_1x_1x_2}\Psi + F_{x_1x_1x_1}\Psi_{x_1} + \Psi_{x_2}, \\ \Psi_{x_1x_2} &= F_{x_1x_2x_2}\Psi + F_{x_1x_1x_2}\Psi_{x_1}, \\ \Psi_{x_2x_2} &= F_{x_2x_2x_2}\Psi + F_{x_1x_2x_2}\Psi_{x_1}. \end{aligned} \tag{114}$$

Deformations of the structure constants generated by the DDA (47) in the WDVV gauge are governed by the system (104) or (106) with the dropped difference terms $\Delta_j A$, etc, i.e. by the system

$$\begin{aligned} &V_{11} + BU_{12} - U_2 B_2 - V_2 = 0, \\ &U_1 V_{12} - V_1 U_{22} = 0, \\ &U_{11}U_{112} - U_2 U_{12} - V_{11}B_{12} + BV_{12} = 0. \end{aligned} \tag{115}$$

Introducing W such that $B = W_2$, one rewrites this system as

$$\begin{aligned} &V_{11} = (UW_2 + V - WU_1)_2, \qquad \left(\frac{V}{U}\right)_1 = \left(\frac{V_1}{U_2}\right)_2, \\ &(U_1 U_{12} - V_1 W_{22})_1 = (UU_2 - WV_1)_2. \end{aligned} \tag{116}$$

7. Deformations generated by three-dimensional Lie algebras and discrete mappings

Deformation moduli for the three-dimensional associative algebras considered in sections 5 and 6 are two dimensional. They are parametrized by two discrete variables x^1 and x^2. One-dimensional deformation moduli arise naturally if one chooses a three-dimensional Lie algebra as the DDA. Among nine such nonequivalent Lie algebras, only three generate discrete deformations [24].

The first such DDA (L_{2b}) is defined by the commutation relations

$$[p_1, p_2] = 0, \qquad [p_1, x] = p_1, \qquad [p_2, x] = 0. \tag{117}$$

Relations (117) imply that

$$[p_1, \varphi(x)] = (T - 1)\varphi(x) \cdot p_1, \qquad [p_2, \varphi(x)] = 0 \tag{118}$$

where $T\varphi(x) = \varphi(x + 1)$. Using this identity, one gets the CS

$$C_1 T C_2 = C_2 C_1. \tag{119}$$

For nondegenerate matrix C_1, one has

$$T C_2 = C_1^{-1} C_2 C_1. \tag{120}$$

The CS (120) is the discrete version of the Lax equation and has similar properties. It has three independent first integrals:

$$I_1 = \operatorname{tr} C_2, \qquad I_2 = \tfrac{1}{2}\operatorname{tr}(C_2)^2, \qquad I_3 = \tfrac{1}{3}\operatorname{tr}(C_2)^3 \tag{121}$$

and represents itself the compatibility condition for the linear problems

$$\Phi C_2 = \lambda \Phi, \qquad T\Phi = \Phi C_1. \tag{122}$$

Note that $\det C_2$ is the first integral too.

The CS (119) is the discrete dynamical system in the space of the structure constants. For the two-dimensional algebra A without the unite element, i.e. when $A = D = L = 0$, it has the form

$$\begin{aligned} &BTE + ETG = EB + MC, \\ &BTM + ETN = E^2 + MG, \\ &CTE + GTG = BG + CN, \\ &CTM + GTN = EG + NG \end{aligned} \tag{123}$$

where B and C are arbitrary functions. For the nondegenerate matrix C_1, i.e. at $BG - CE \neq 0$, in the resolved form it is

$$\begin{aligned} TE &= \frac{GM - EN}{BG - CE} C, \qquad & TG &= B + \frac{BN - CM}{BG - CE} C, \\ TM &= \frac{GM - EN}{BG - CE} G, \qquad & TN &= E + \frac{BN - CM}{BG - CE} G. \end{aligned} \tag{124}$$

This system defines discrete deformations of the structure constants. Formally, these deformations can be considered as the reduction $T_2 =$ identity of the general case (90).

The second example has given the solvable DDA (L_4) defined by the commutation relations

$$[p_1, p_2] = 0, \qquad [p_1, x] = p_1, \qquad [p_2, x] = p_2. \tag{125}$$

For this DDA, one has

$$[p_j, \varphi(x)] = (T - 1)\varphi(x) p_j, \qquad j = 1, 2, \tag{126}$$

where $\varphi(x)$ is an arbitrary function and T is the shift operator $T\varphi(x) = \varphi(x+1)$. With the use of (126), one arrives at the following CS:

$$C_1TC_2 = C_2TC_1. \tag{127}$$

For nondegenerate matrix C_1, equation (127) is equivalent to the equation $T\left(C_2C_1^{-1}\right) = C_1^{-1}C_2$ or

$$TU = C_1^{-1}UC_1 \tag{128}$$

where $U \doteqdot C_2C_1^{-1}$. Using this form of the CS, one promptly concludes that the CS (85) has three independent first integrals:

$$I_1 = \mathrm{tr}\left(C_2C_1^{-1}\right), \qquad I_2 = \frac{1}{2}\mathrm{tr}\left(C_2C_1^{-1}\right)^2, \qquad I_3 = \frac{1}{3}\mathrm{tr}\left(C_2C_1^{-1}\right)^3 \tag{129}$$

and is representable as the commutativity condition for the linear system

$$\Phi C_2C_1^{-1} = \lambda\Phi, \qquad T\Phi = \Phi C_1. \tag{130}$$

For the two-dimensional algebra A without unite element, the CS (127) is the system of four equations for six functions

$$\begin{aligned} &BTE + ETG = ETB + MTC, \\ &BTM + ETN = ETE + MTG, \\ &CTE + GTG = GTB + NTC, \\ &CTM + GTN = GTE + NTG. \end{aligned} \tag{131}$$

Choosing B and C as free functions and assuming that $BG - CE \neq 0$, one can easily resolve (131) with respect to TE, TG, TM, TN. For instance, with $B = C = 1$, one gets the following four-dimensional mapping:

$$\begin{aligned} &TE = M - E\frac{M-N}{E-G}, \qquad TG = 1 + \frac{M-N}{E-G}, \\ &TM = N + (N-G)\frac{M-N}{E-G} - G\left(\frac{M-N}{E-G}\right)^2, \\ &TN = M + (1-E)\frac{M-N}{E-G} + \left(\frac{M-N}{E-G}\right)^2. \end{aligned} \tag{132}$$

In a similar manner, one finds the CS associated with the DDA (L_5) defined by the relations

$$[p_1, p_2] = 0, \qquad [p_1, x] = p_1, \qquad [p_2, x] = -p_2. \tag{133}$$

Since in this case

$$[p_1, \varphi(x)] = (T-1)\varphi(x)p_1, \qquad [p_2, \varphi(x)] = (T^{-1}-1)\varphi(x)p_2 \tag{134}$$

the CS takes the form

$$C_1TC_2 = C_2T^{-1}C_1. \tag{135}$$

For nondegenerate C_2, it is equivalent to

$$TV = C_2VC_2^{-1} \tag{136}$$

where $V \doteqdot T^{-1}C_1 \cdot C_2$. Similar to the previous case, the CS has three first integrals

$$I_1 = \mathrm{tr}(C_1TC_2), \qquad I_2 = \frac{1}{2}\mathrm{tr}(C_1TC_2)^2, \qquad I_3 = \frac{1}{3}\mathrm{tr}(C_1TC_2)^3 \tag{137}$$

and is equivalent to the compatibility condition for the linear system

$$(T^{-1}C_1)C_2\Phi = \lambda\Phi, \qquad T\Phi = C_2\Phi. \tag{138}$$

Note that the CS (135) is of the form (70) with $T_1 = T$, $T_2 = T^{-1}$. Thus, the deformations generated by L_5 can be considered as the reductions of the discrete deformations (70) under the constraint $T_1T_2C_{jk}^n = C_{jk}^n$.

8. Discrete oriented associativity equation

For general discrete deformations , similar to the quantum deformations [13], the global associativity condition $[C_j, C_k] = 0$ is not preserved for all values of the deformation parameters. Deformations of associative algebras for which the associativity condition is globally valid (isoassociative deformations) form an important class of all possible deformations [12–21]. Within the theories of Frobenius and F-manifolds [12–14] and also for the coisotropic and quantum deformations [19, 21] such deformations are characterized by the existence of a set of functions $\Phi^l, l = 1, \ldots, N$, such that

$$C^l_{jk} = \frac{\partial^2\Phi^l}{\partial x^j \partial x^k}, \qquad j,k,l = 1,\ldots,N. \tag{139}$$

These functions obey the oriented associativity equation [87, 13]

$$\sum_m \frac{\partial^2\Phi^n}{\partial x^l \partial x^m}\frac{\partial^2\Phi^m}{\partial x^j \partial x^k} - \sum_m \frac{\partial^2\Phi^n}{\partial x^j \partial x^m}\frac{\partial^2\Phi^m}{\partial x^l \partial x^k} = 0, \qquad j,k,l,n = 1,\ldots,N. \tag{140}$$

Here we will present discrete versions of this equation. So, we consider the N-dimensional associative algebra A, deformations generated by the DDA (48) and restrict ourselves to the isoassociative deformations for which

$$[C_j(x), C_k(x)] = 0, \qquad j,k = 1,\ldots,N. \tag{141}$$

A class of solutions of the CS (70) equations is given by the formula

$$C_j = g^{-1}\Delta_j g \tag{142}$$

where $g(x)$ is a matrix-valued function. Since $C^l_{jk} = C^l_{kj}$, one has

$$\Delta_j g^n_k = \Delta_k g^n_j, \qquad j,k,n = 1,\ldots,N, \tag{143}$$

where g^n_k are matrix elements of g. Hence

$$g^n_k = g^n_{0k} + \alpha\Delta_k\Phi^n, \qquad k,n = 1,\ldots,N, \tag{144}$$

where g^n_{0k} and α are arbitrary constants and Φ^n are functions. Substitution of (142) and (144) into (141) gives

$$\sum_{m,t} \Delta_l\Delta_t\Phi^n \cdot (g^{-1})^t_m \Delta_j\Delta_k\Phi^m - \sum_{m,t} \Delta_l\Delta_t\Phi^n \cdot (g^{-1})^t_m \Delta_j\Delta_k\Phi^m = 0. \tag{145}$$

Since in the continuous limit $\Delta_j \to \varepsilon\frac{\partial}{\partial x^j}$, $g^n_{0k} = \delta^n_k$, $\alpha = 0$, $\varepsilon \to 0$ the system (145) is reduced to (140), it represents a discrete isoassociative version of the oriented associativity equation.

For the DDA (47), the CS (69) has a solution $C_j = g^{-1}T_j g$ and symmetry of C implies that $g^n_k = T_k\Phi^n$ where Φ^n are functions. Substitution of this expression into the associativity condition (141) gives

$$\sum_{m,t}(T_jT_m\Phi^n)(g^{-1})^m_t T_kT_l\Phi^t = \sum_{m,t}(T_kT_m\Phi^n)(g^{-1})^m_t T_jT_l\Phi^t. \tag{146}$$

Different discrete version of equation (140) arises if one relaxes the condition (141) and requires that the following quasi-associativity condition:

$$C_lT_lC_j = C_jT_jC_l, \qquad j,l = 1,\ldots,N, \tag{147}$$

is valid for all values of deformation parameters. In this case, the CS (70) is reduced to the system

$$\Delta_l C_j - \Delta_j C_l = 0, \qquad j,l = 1,\ldots,N, \tag{148}$$

which implies the existence of the matrix-valued function Φ such that

$$C_j = \Delta_j \Phi, \qquad j = 1, \ldots, N.$$

Since $C^l_{jk} = \Delta_j \Phi^l_k = C^l_{kj}$, one has

$$\Phi^l_k = \Delta_k \Phi^l, \qquad l, k = 1 \ldots, N,$$

where Φ^l, $l = 1, \ldots, N$ are functions. So

$$C^l_{jk} = \Delta_j \Delta_k \Phi^l.$$

Finally, the quasi-associativity condition (147) takes the form

$$\sum_m \Delta_j \Delta_k T_l \Phi^m \cdot \Delta_l \Delta_m \Phi^n - \sum_m \Delta_l \Delta_k T_j \Phi^m \cdot \Delta_j \Delta_m \Phi^n = 0, \qquad j, k, l, n = 1, \ldots, N, \tag{149}$$

which is a discrete version of the oriented associativity equation (140). Any solution of the systems (145), (146) and (149) defines discrete deformation of the structure constants C^l_{jk}.

9. Discrete deformations for a class of special associative algebras

Several important discrete integrable equations arise as the CSs governing discrete deformations of a class of associative algebras for which the multiplication of only distinct elements of the basis is defined. For such algebras, the table of multiplication is of the form

$$\mathbf{P}_j \mathbf{P}_k = A_{jk} \mathbf{P}_j + B_{jk} \mathbf{P}_k + C_{jk} \mathbf{P}_0, \qquad j \neq k, \qquad j, k = 1, 2, \ldots, N. \tag{150}$$

The commutativity of the basis implies that $A_{kj} = B_{jk}, C_{jk} = C_{kj}$. The algebra of the functions $f_j = \frac{a_j}{\lambda - \lambda_j}$ with simple poles in distinct points is an example of such algebra (see (36)).

We choose the algebra (47) as DDA. Recall that the algebra of shifts $p_j = T_j$ is a realization of this DDA. Equations (62) and (45) in this case take the form

$$p_j p_k |\Psi\rangle = A_{jk} p_j |\Psi\rangle + B_{jk} p_k |\Psi\rangle + C_{jk} |\Psi\rangle, \qquad j \neq k, \tag{151}$$

and

$$((p_j p_k) p_l - p_j (p_k p_l)) |\Psi\rangle = \sum_t \Omega^t_{klj} p_t |\Psi\rangle, \qquad j \neq k \neq l \neq j, \tag{152}$$

respectively. The corresponding CS is given by the following system of equations:

$$A_{kl} T_l B_{jk} = B_{jk} T_j A_{kl}, \tag{153}$$

$$B_{lj} T_l A_{jk} = A_{jk} T_j A_{kl} + A_{jl} T_j B_{kl} + T_j C_{kl}, \tag{154}$$

$$B_{jl} T_j B_{kl} = A_{lj} T_l A_{jk} + A_{lk} T_l B_{jk} + T_l C_{jk}, \tag{155}$$

$$C_{lj} T_l A_{jk} + C_{lk} T_l B_{jk} = C_{jk} T_j A_{kl} + C_{jl} T_j B_{kl} \tag{156}$$

where all indices are distinct.

Denoting $\Phi_j \doteqdot T_j |\Psi\rangle$, $\Phi_{jk} \doteqdot T_j T_k |\Psi\rangle$, one rewrites (151) as the system

$$\Phi_{jk} = A_{jk} \Phi_j + B_{jk} \Phi_k + C_{jk} \Phi, \qquad j \neq k. \tag{157}$$

This system can be treated as the set of relations between the points Φ, Φ_j, Φ_{jk} connected by the shifts $\Phi_j = T_j \Phi$, $\Phi_{jk} = T_j T_k \Phi$.

The CS (153)–(156) and equations (151), (157) are invariant under the gauge transformations (84) for which $|\Psi\rangle \to |\widetilde{\Psi}\rangle = g^{-1}|\Psi\rangle$, $\Phi \to \widetilde{\Phi} = g^{-1}\Phi$ and

$$A_{jk} \to \widetilde{A}_{jk} = \frac{g_j}{g_{jk}} A_{jk}, \qquad B_{jk} \to \widetilde{B}_{jk} = \frac{g_k}{g_{jk}} B_{jk}, \qquad C_{jk} \to \widetilde{C}_{jk} = \frac{g}{g_{jk}} C_{jk}. \tag{158}$$

Family of the structure constants connected by the transformations with different g form orbits of gauge equivalent structure constants. These orbits are characterized by the $N(N-1)$ gauge invariants

$$I^1_{jk} = \frac{B_{jk} \cdot T_j^{-1} A_{jk}}{A_{jk} \cdot T_k^{-1} B_{jk}}, \qquad I^2_{jk} = \frac{C_{jk}}{B_{jk} \cdot T_j^{-1} A_{jk}}, \qquad j \succ k,\ j,k = 1,\dots,N, \tag{159}$$

or

$$\widetilde{I}^1_{jk} = \frac{B_{jk} \cdot T_j^{-1} A_{jk}}{C_{jk}}, \qquad \widetilde{I}^2_{jk} = \frac{A_{jk} \cdot T_k^{-1} B_{jk}}{C_{jk}}, \qquad j \succ k. \tag{160}$$

These invariants are the discrete version of the well-known gauge invariants for the standard Laplace equations (see e.g. [88]). The invariants (160) coincide with those introduced earlier in [89] up to the trivial redefinitions.

Similar to the continuous case (see [88]), these invariants have also the meaning of the defects of the factorizability of the discrete operators, namely

$$\begin{aligned} T_j T_k - A_{jk} T_j - B_{jk} T_k - C_{jk} &= (T_j - B_{jk})\big(T_k - \big(T_j^{-1} A_{jk}\big)\big) + C_{jk}\big(\widetilde{I}^1_{jk} - 1\big) \\ &= (T_j - A_{jk})\big(T_k - \big(T_k^{-1} B_{jk}\big)\big) + C_{jk}\big(\widetilde{I}^2_{jk} - 1\big). \end{aligned} \tag{161}$$

The orbits of gauge transformations (158) have one distinguished element. Indeed, it is easy to see that choosing $g = \widehat{\Phi}$ where $\widehat{\Phi}$ is a solution of the system (157), one gets the structure constants $\widetilde{A}_{jk}$, $\widetilde{B}_{jk}$, $\widetilde{C}_{jk}$ which obey the relation

$$\widetilde{A}_{jk} + \widetilde{B}_{jk} + \widetilde{C}_{jk} = 1, \qquad j \neq k. \tag{162}$$

In this gauge relations (157) and corresponding configuration of the points are invariant under translations in space.

The multidimensional multiplication table (150) and the CS (153)–(156) are, in fact, the systems of three-dimensional irreducible subsystems glued all together. Indeed, for any three distinct indices j, k, l the corresponding subtable (150) and subsystem of (153)–(156) are closed with other deformation variables playing the role of parameters. So, the study of the CS (153)–(156) is reduced basically to the study of the three-dimensional case. Choosing any three distinct indices j, k, l and denoting them 1, 2, 3 respectively, we present a corresponding subtable of multiplication as

$$\mathbf{P}_1\mathbf{P}_2 = A\mathbf{P}_1 + B\mathbf{P}_2 + L\mathbf{P}_0, \tag{163}$$

$$\mathbf{P}_1\mathbf{P}_3 = C\mathbf{P}_1 + D\mathbf{P}_3 + M\mathbf{P}_0, \tag{164}$$

$$\mathbf{P}_2\mathbf{P}_3 = E\mathbf{P}_2 + G\mathbf{P}_3 + N\mathbf{P}_0. \tag{165}$$

The subsystem of the CS for the structure constants $A, B, \dots, N$ is given by the system of equations

$$\frac{A_3}{A} = \frac{C_2}{C}, \qquad \frac{B_3}{B} = \frac{E_1}{E}, \qquad \frac{D_2}{D} = \frac{G_1}{G}, \tag{166}$$

$$(A_3 - E_1)L + B_3 N - G_1 M = 0, \qquad (A_3 - G_1)D - E_1 A - N_1 = 0, \tag{167}$$

$$(A_3 - G_1)D + B_3 G + L_3 = 0, \qquad (C_2 - E_1)L + D_2 N - G_1 M = 0, \tag{168}$$

$$(C_2 - E_1)B + D_2E + M_2 = 0, \qquad C_2L - A_3M + (D_2 - B_3)N = 0 \tag{169}$$

where we denote $A_j = T_jA$, $B_j = T_jB$ and so on. For the distinguished gauge (162) one has

$$A + B + L = 1, \tag{170}$$

$$C + D + M = 1, \tag{171}$$

$$E + G + N = 1. \tag{172}$$

One may note the coincidence of these relations with those which define the so-called barycentric coordinates of a point in the plane of a given triangle (see e.g. [90, 91]).

10. Menelaus relation as an associativity condition and Menelaus deformations

Deformations of the associative algebras of the type (150) but without unite element are of particular interest. In this case, the table of multiplication is given by (15) with $C_{jk} = 0$. As in the previous section, we choose the algebra (47) as DDA.

For the three-dimensional irreducible subalgebra, the multiplication table is given by (163–165) with $L = M = N = 0$ while equations (153) are of the form

$$\Phi_{12} = A\Phi_1 + B\Phi_2, \qquad \Phi_{13} = C\Phi_1 + D\Phi_3, \qquad \Phi_{23} = E\Phi_2 + G\Phi_3. \tag{173}$$

The distinguished gauge is then defined by the relations

$$A + B = 1, \qquad C + D = 1, \qquad E + G = 1. \tag{174}$$

The associativity conditions

$$\mathbf{P}_1(\mathbf{P}_2\mathbf{P}_3) = \mathbf{P}_2(\mathbf{P}_3\mathbf{P}_1) = \mathbf{P}_3(\mathbf{P}_1\mathbf{P}_2) \tag{175}$$

in this case are equivalent to the system

$$(A - G)C - EA = 0, \qquad (A - G)D + BG = 0, \qquad (C - E)B + DE = 0. \tag{176}$$

This system is a rather special one. First, equations (176) imply that $AED + BCG = 0$. Then it is easy to check that one has the following.

Proposition 3. *For nonvanishing $A, B, \ldots, G$, the associativity conditions (176) are equivalent to the equation*

$$AED + BCG = 0 \tag{177}$$

and one of equations (176), for instance, the equation

$$(A - G)C - EA = 0. \tag{178}$$

This form of associativity conditions is of interest for several reasons. One of them is connected with gauge transformations. It was noted in the previous section that the associativity conditions and, in particular, conditions (176) are not invariant under general gauge transformations. The form (177), (178) of associativity conditions has a special property. Namely, due to the relation

$$\widetilde{A}\widetilde{E}\widetilde{D} + \widetilde{B}\widetilde{C}\widetilde{G} = \frac{g_1g_2g_3}{g_{12}g_{23}g_{13}}(AED + BCG) \tag{179}$$

the condition (177) is invariant under gauge transformations. So, this relation is a characteristic one for the orbits of gauge equivalent structure constants.

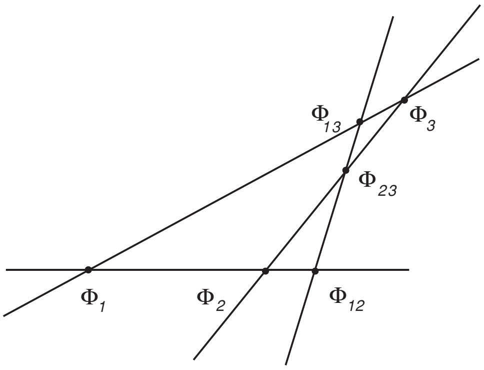

Figure 1. Menelaus configuration.

Moreover, one can show that the condition (177) guarantees that the set of constants A, $B, \ldots, G$ can be converted into the set of constants $\widetilde{A}, \widetilde{B}, \ldots, \widetilde{G}$ obeying the associativity conditions (176) by the gauge transformation with $g = \widehat{\Phi}$ where $\widehat{\Phi}$ is a solution of equations (173).

Then, if one treats equations (173) as the relations between six points $\Phi_1, \Phi_2, \Phi_3, \Phi_{12}, \Phi_{23}, \Phi_{13}$ on the (complex) plane, then the lhs of (177) coincides with the determinant of the matrix of transformation from the set of three points Φ_1, Φ_2, Φ_3 to the set $\Phi_{12}, \Phi_{23}, \Phi_{13}$. So, for associative algebras such transformation is singular.

The relation between six points $\Phi_1, \Phi_2, \Phi_3, \Phi_{12}, \Phi_{23}, \Phi_{13}$ defined by (173) has really a remarkable geometrical meaning in the distinguished gauge (176). First, relations (173), in virtue of the conditions (174), mean that three points $\Phi_1, \Phi_2, \Phi_{12}$ are collinear as well as the sets of points $\Phi_1, \Phi_3, \Phi_{13}$ and $\Phi_2, \Phi_3, \Phi_{23}$. Then relations (177), (178) imply that the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ are collinear too, i.e.

$$\Phi_{12} = \frac{A}{C}\Phi_{13} + \frac{B}{E}\Phi_{23} \tag{180}$$

with $\frac{A}{C} + \frac{B}{E} = 1$. Thus, in the gauge (174) relations (173) describe the set of four triples $(\Phi_1, \Phi_2, \Phi_{12}), (\Phi_1, \Phi_3, \Phi_{13}), (\Phi_2, \Phi_3, \Phi_{23})$ and $(\Phi_{12}, \Phi_{13}, \Phi_{23})$ of collinear points. It is nothing but the celebrated Menelaus configuration of the classical geometry (figure 1) (see e.g. [90, 91]). Due to (174), relations (173) and Menelaus configuration are translationally invariant.

Relations (173) and (174) allow us to express $A, B, \ldots, G$ in terms of Φ. One gets

$$\begin{aligned} A &= \frac{\Phi_{12}^M - \Phi_2^M}{\Phi_1^M - \Phi_2^M}, & B &= -\frac{\Phi_{12}^M - \Phi_1^M}{\Phi_1^M - \Phi_2^M}, & C &= \frac{\Phi_{13}^M - \Phi_3^M}{\Phi_1^M - \Phi_3^M}, \\ D &= -\frac{\Phi_{13}^M - \Phi_1^M}{\Phi_1^M - \Phi_3^M}, & E &= \frac{\Phi_{23}^M - \Phi_3^M}{\Phi_2^M - \Phi_3^M}, & G &= -\frac{\Phi_{23}^M - \Phi_2^M}{\Phi_2^M - \Phi_3^M} \end{aligned} \tag{181}$$

where we denote by Φ^M the solution of the system (173), (174). In such a parametrization of $A, B, \ldots, G$, relations (177), (178) are equivalent to the single equation

$$\frac{\left(\Phi_1^M - \Phi_{12}^M\right)\left(\Phi_2^M - \Phi_{23}^M\right)\left(\Phi_3^M - \Phi_{13}^M\right)}{\left(\Phi_{12}^M - \Phi_2^M\right)\left(\Phi_{23}^M - \Phi_3^M\right)\left(\Phi_{13}^M - \Phi_1^M\right)} = -1. \tag{182}$$

It is the famous Menelaus relation (see [90, 91]) which guarantees that for any three points Φ_1, Φ_2, Φ_3 on the plane the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ are collinear. In our formulation, the

Menelaus relation (182) is nothing else than the associativity conditions (177), (178) written in terms of Φ^M. Thus, the Menelaus theorem is intimately connected with the associative algebra (163), (165) with $L = M = N = 0$ in the distinguished gauge (174). Amazingly, this interpretation seems to be not that distant from the old-known algebraic proofs of the Menelaus theorem where the relation of the type (177) already has appeared [90].

Comparing equations (182) and (26), one concludes that the Menelaus relation and NSF for the Schwarzian KP hierarchy coincide for given six points Φ_1, Φ_2, Φ_3, Φ_{12}, Φ_{23}, Φ_{13} [48]. In the paper [48], this coincidence has been extended to the discrete equation thus defining the Menelaus lattice.

In our approach, discrete deformations of the algebra (163)–(166) and, hence, of the Menelaus configuration is governed by the CS

$$\frac{A_3}{A} = \frac{C_2}{C}, \qquad \frac{B_3}{B} = \frac{E_1}{E}, \qquad \frac{D_2}{D} = \frac{G_1}{G}, \tag{183}$$

$$(A_3 - G_1)C - E_1 A = 0, \tag{184}$$

$$(A_3 - G_1)D + B_3 G = 0, \tag{185}$$

$$(C_2 - E_1)B + D_2 E = 0. \tag{186}$$

Equations (184), (185) imply that

$$AE_1 D + B_3 CG = 0,$$

which is equivalent to relation (177) in virtue of the second equation (183). Thus, the Menelaus relation (182) is preserved by deformations. Discrete deformations of the Menelaus configuration given by equations (183)–(186) generate a lattice on the plane. It is a straightforward check that equations (183)–(186) rewritten in terms of the 'shape' parameters $\alpha = -\frac{B}{A}, \beta = -\frac{G}{E}, \gamma = -\frac{C}{D}$ coincide with those derived in [48] (theorem 3). Thus, the Menelaus lattice represents a realization of discrete deformations of the associative algebra (163)–(165) with $L = M = N = 0$ in the gauge (174).

In [48] it was shown that under the constraint $\Phi_{23} = \Phi$ the Menelaus relation (182) is reduced to the Schwarzian discrete KdV equation (9). So, the under this constraint one has a subclass of deformations governed by the discrete KdV equation (9).

11. KP configurations, discrete KP deformations and their gauge equivalence to Menelaus configurations

There is an another distinguished gauge on the orbits under consideration. It is given by the relations

$$A + B = 0, \qquad C + D = 0, \qquad E + G = 0 \tag{187}$$

for which equations (173) take the form

$$\Phi_{12}^{KP} = A\left(\Phi_1^{KP} - \Phi_2^{KP}\right), \qquad \Phi_{13}^{KP} = C\left(\Phi_1^{KP} - \Phi_3^{KP}\right), \qquad \Phi_{23}^{KP} = E\left(\Phi_2^{KP} - \Phi_3^{KP}\right). \tag{188}$$

We shall refer to this gauge as the KP gauge for the reason which will be clarified below. In this gauge, relation (177) becomes a trivial identity and, hence, the associativity conditions are reduced to the single equation

$$AC + EC - AE = 0. \tag{189}$$

Proposition 4. *In the KP gauge (187), the CS (183)–(186) is equivalent to the associativity condition (189) and equations*

$$\frac{A_3}{A}=\frac{C_2}{C}=\frac{E_1}{E}. \tag{190}$$

Proof. In the gauge (187), equations (183) are reduced to equations (190) while equations (184)–(186) are equivalent to the single equation $A_3C+E_1C-AE_1=0$. Due to (190), this equation is equivalent to the associativity condition (189). □

Equations (190) imply the existence of a function τ such that

$$A=-\frac{\tau_1\tau_2}{\tau\tau_{12}},\qquad C=-\frac{\tau_1\tau_3}{\tau\tau_{13}},\qquad E=-\frac{\tau_2\tau_3}{\tau\tau_{23}}. \tag{191}$$

Substitution of these expressions into (189) gives the Hirota bilinear equation for the KP hierarchy, i.e.

$$\tau_1\tau_{23}-\tau_2\tau_{13}+\tau_3\tau_{12}=0. \tag{192}$$

This fact justifies the name of the gauge (187). We would like to emphasize that the Hirota–Miwa equation (192) is nothing but the associativity condition (189) with the structure constants A, C, E parametrized by τ-function.

Equations (188) with A, C, E of the form (191) coincide with well-known linear problems for the Hirota–Miwa bilinear equation [92, 35]. The parametrization (191) suggests to rewrite equation (188) in the gauge equivalent form

$$\widehat{\Phi}^{KP}_{12}=\frac{\tau_1}{\tau}\widehat{\Phi}^{KP}_2-\frac{\tau_2}{\tau}\widehat{\Phi}^{KP}_1,\quad \widehat{\Phi}^{KP}_{13}=\frac{\tau_1}{\tau}\widehat{\Phi}^{KP}_3-\frac{\tau_3}{\tau}\widehat{\Phi}^{KP}_1,\quad \widehat{\Phi}^{KP}_{23}=\frac{\tau_2}{\tau}\widehat{\Phi}^{KP}_3-\frac{\tau_3}{\tau}\widehat{\Phi}^{KP}_2, \tag{193}$$

where $\widehat{\Phi}^{KP}=\tau\Phi^{KP}$. The condition of compatibility for the system (193) is equivalent to equation (192) too. Equations (193) also imply that

$$\widehat{\Phi}^{KP}_1\widehat{\Phi}^{KP}_{23}-\widehat{\Phi}^{KP}_2\widehat{\Phi}^{KP}_{13}+\widehat{\Phi}^{KP}_3\widehat{\Phi}^{KP}_{12}=0, \tag{194}$$

which coincides with equation (192). So, one can choose $\widehat{\Phi}^{KP}=\widehat{\tau}$ where $\widehat{\tau}$ is a solution of the Hirota–Miwa equation (192). Thus, one has

$$\Phi^{KP}=\frac{\widehat{\tau}}{\tau}, \tag{195}$$

which is again the well-known formula from the theory of the KP hierarchy.

Geometrical configuration on the plane formed by six points $\Phi^{KP}_1, \Phi^{KP}_2, \Phi^{KP}_3, \Phi^{KP}_{12}, \Phi^{KP}_{23}, \Phi^{KP}_{13}$ with real A, C, E is of interest too. We first observe that the points $\Phi^{KP}_{12}, \Phi^{KP}_{23}, \Phi^{KP}_{13}$ lie on the straight lines passing through the origin 0 and parallel to the straight lines passing through the points $(\Phi^{KP}_1, \Phi^{KP}_2)$, $(\Phi^{KP}_1, \Phi^{KP}_3)$, $(\Phi^{KP}_2, \Phi^{KP}_3)$, respectively. Then, due to the associativity condition (189), the points $\Phi^{KP}_{12}, \Phi^{KP}_{23}, \Phi^{KP}_{13}$ are collinear. Indeed, equations (188) imply that

$$\frac{1}{C}\Phi^{KP}_{13}-\frac{1}{A}\Phi^{KP}_{12}-\frac{1}{E}\Phi^{KP}_{23}=0 \tag{196}$$

while relation (189) is equivalent to the condition $\frac{1}{C}-\frac{1}{A}-\frac{1}{E}=0$.Thus, the points $\Phi^{KP}_1, \Phi^{KP}_2, \Phi^{KP}_3, \Phi^{KP}_{12}, \Phi^{KP}_{23}, \Phi^{KP}_{13}$ form the KP configuration on the complex plane shown in figure 2.

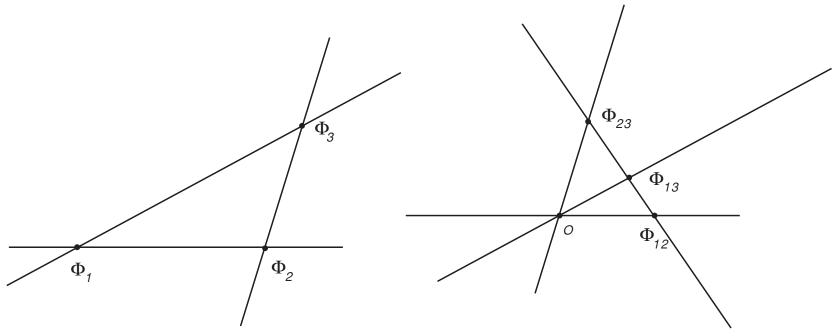

Figure 2. KP configuration.

The associativity condition (189) provides us also with the relation between the directed lengths for the KP configuration. Indeed, expressing A, C, E from (188) in terms of Φ^{KP} and substituting into (189), one gets

$$\frac{\Phi_1^{KP}-\Phi_2^{KP}}{\Phi_{12}^{KP}}+\frac{\Phi_2^{KP}-\Phi_3^{KP}}{\Phi_{23}^{KP}}+\frac{\Phi_3^{KP}-\Phi_1^{KP}}{\Phi_{31}^{KP}}=0. \tag{197}$$

Since for real A, C, E $\frac{\Phi_1^{KP}-\Phi_2^{KP}}{\Phi_{12}^{KP}}=\frac{\left|\Phi_1^{KP}-\Phi_2^{KP}\right|}{\left|\Phi_{12}^{KP}\right|}$, etc, formula (197) represents the relation between the directed lengths $\left|\Phi_1^{KP}-\Phi_2^{KP}\right|$ of the interval (Φ_1, Φ_2), etc. In contrast to the Menelaus case, relations (188), (197) and KP configuration are not invariant with respect to the displacements on the plane.

The Menelaus and KP configurations look quite different. For instance, the points $\Phi_1, \Phi_2, \Phi_{12}$, etc are collinear in the Menelaus case and they are not in the KP case. Nevertheless, they are closely connected, namely they are gauge equivalent to each other. To demonstrate this, let us consider the gauge transformation $\Phi^{KP}\to\widetilde{\Phi}^{KP}=g^{-1}\Phi^{KP}$. Under this transformation

$$\begin{aligned}
&\widetilde{A}=\frac{g_1}{g_{12}}A^{KP}, && \widetilde{B}=-\frac{g_2}{g_{12}}A^{KP}, && \widetilde{C}=\frac{g_1}{g_{13}}C^{KP},\\
&\widetilde{D}=-\frac{g_3}{g_{13}}C^{KP}, && \widetilde{E}=\frac{g_2}{g_{23}}E^{KP}, && \widetilde{G}=-\frac{g_3}{g_{23}}E^{KP}
\end{aligned} \tag{198}$$

and

$$\widetilde{A}+\widetilde{B}=\frac{g_1-g_2}{g_{12}}A^{KP}, \qquad \widetilde{C}+\widetilde{D}=\frac{g_1-g_3}{g_{13}}C^{KP}, \qquad \widetilde{E}+\widetilde{G}=\frac{g_2-g_3}{g_{23}}E^{KP}.$$

Choosing $g=\widehat{\Phi}^{KP}$ where $\widehat{\Phi}^{KP}$ is a solution of equations (188) with the same A^{KP}, C^{KP}, E^{KP}, one gets

$$\widetilde{A}+\widetilde{B}=1, \qquad \widetilde{C}+\widetilde{D}=1, \qquad \widetilde{E}+\widetilde{G}=1. \tag{199}$$

Thus, $\widetilde{\Phi}^{KP}=\frac{\Phi^{KP}}{\widehat{\Phi}^{KP}}=\Phi^M$ and so the KP configuration is converted into the Menelaus configuration. In geometric terms, this gauge transformation is the local (depending on the point) homothetic transformation. So, in order to construct the Menelaus configuration out of the KP we need two KP configurations with the same A, C, E. Similarly, formula (195) shows us that to construct KP configuration one needs two configurations of six points defined by the Hirota–Miwa equation (192).

Another way to demonstrate the gauge equivalence between Menelaus and KP configurations is based on formula (27). Eliminating Ψ^*, one can rewrite this equation as [65]

$$\frac{\widetilde{\Delta}_j \Phi}{\widetilde{\Delta}_k \Phi} = \frac{T_j^{-1}\Psi}{T_k^{-1}\Psi}, \qquad j \neq k,\ j, k = 1, 2, 3, \tag{200}$$

where $\widetilde{\Delta}_j = T_j^{-1} - 1$. These equations imply the following:

$$\Phi_{jk} = -\frac{\Psi_k}{\Psi_j - \Psi_k}\Phi_j + \frac{\Psi_j}{\Psi_j - \Psi_k}\Phi_k, \qquad j \neq k,\ j, k = 1, 2, 3. \tag{201}$$

Thus, $\Phi = \Phi^M$ and formulae (201) give us the parametrization of $A, B, \ldots, G$ in terms of Ψ in the Menelaus gauge.

Performing the gauge transformation $\Phi = \Psi\widetilde{\Phi}$, one gets the equations

$$\widetilde{\Phi}_{jk} = -\frac{\Psi_j\Psi_k}{\Psi_{jk}(\Psi_j - \Psi_k)}\widetilde{\Phi}_j + \frac{\Psi_j\Psi_k}{\Psi_{jk}(\Psi_j - \Psi_k)}\widetilde{\Phi}_k, \qquad j \neq k,\ j, k = 1, 2, 3. \tag{202}$$

So, $\widetilde{\Phi} = \Phi^{KP}$ and formulae (202) provide us with the parametrization of $A, B, \ldots, G$ in the KP case in terms of Ψ.

It is an easy check that the ratio $\frac{\widetilde{\Phi}}{\Phi}$ of two solutions of the system (202) obeys the Menelaus system (201) with $\Psi = \frac{1}{\Phi}$ in agreement with the formula $\Phi^M = \frac{\Phi^{KP}}{\widetilde{\Phi}^{KP}}$. Then, using parametrization of A_{jk} and B_{jk} given by formulae (201) and (202), it is not difficult to show that the invariants I^1_{jk} (159) are equal in Menelaus and KP gauges.

Note also that in the parametrization (201) of the Menelaus $A, B, \ldots, G$ the associativity conditions (177), (178) are satisfied identically. In the KP case (202), these associativity conditions are satisfied in virtue of equation (197) for $\Phi^{KP} = \frac{1}{\Psi}$. The collinearity conditions for points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ are obviously satisfied in both cases in parametrizations (201) and (202).

At last, the Menelaus figure 1 and KP figure 2 are converted to each other by the gauge transformation (local homothety) $\Phi^M = \Psi\Phi^{KP}$.

12. Multidimensional Menelaus and KP configurations and deformations

Now let us consider the N-dimensional algebra (150) with $C_{jk} = 0$ in the Menelaus gauge $A_{jk} + B_{jk} = 1$, $j \neq k$. The associativity conditions in this case take the form

$$A_{jk}A_{jl} = A_{kl}A_{jk} + A_{jl}B_{kl}, \qquad B_{kl}B_{jl} = A_{jk}A_{lj} + B_{jk}A_{lk} \tag{203}$$

where all indices are distinct. Multiplying the first of these equations by B_{jl}, second by A_{jl} and subtracting, one gets

$$A_{jk}A_{kl}B_{jl} + A_{jl}B_{jk}B_{kl} = 0, \tag{204}$$

with all distinct indices.

For $N = 3$, it is just the Menelaus relation (177). For arbitrary N, the above Menelaus-type relations imply

$$\frac{B_{12}}{A_{12}}\frac{B_{23}}{A_{23}}\frac{B_{34}}{A_{34}}\cdots\frac{A_{1N}}{B_{1N}} = (-1)^N. \tag{205}$$

From equations (157) with $C_{jk} = 0$ in the Menelaus gauge, one gets

$$A_{jk} = \frac{\Phi_{jk} - \Phi_k}{\Phi_j - \Phi_k}, \qquad B_{jk} = -\frac{\Phi_{jk} - \Phi_j}{\Phi_j - \Phi_k}, \qquad j, k = 1, \ldots, N. \tag{206}$$

Substituting these expressions into (205), one obtains the relation

$$\frac{(\Phi_1-\Phi_{12})(\Phi_2-\Phi_{23})\cdots(\Phi_N-\Phi_{1N})}{(\Phi_{12}-\Phi_2)(\Phi_{23}-\Phi_3)\cdots(\Phi_{1N}-\Phi_1)}=(-1)^N. \tag{207}$$

It is the generalization of the Menelaus relation to N-gons on the plane where $\Phi_1,\ldots,\Phi_N$ are vertices of the N-gon and $\Phi_{12},\Phi_{23},\ldots,\Phi_{N1}$ are points of intersections of a straight line with the corresponding sides of N-gon [93] (see also [50]). Again it is just the associativity condition (204).

Deformations of the N-dimensional algebra (150) and N-gon Menelaus configuration are governed by the CS

$$\begin{aligned} A_{kl}T_lB_{jk} &= B_{jk}T_jA_{kl},\\ B_{lj}T_lA_{jk} &= A_{jk}T_jA_{kl}+A_{jl}T_jB_{kl},\\ B_{jl}T_jB_{kl} &= A_{lj}T_lA_{jk}+A_{lk}T_lB_{jk}, \end{aligned} \tag{208}$$

with all distinct indices j,k,l.

In the KP gauge $A_{jk}+B_{jk}=0$ the associativity conditions for the N-dimensional algebra (150) with $C_{jk}=0$ are reduced to the system

$$A_{jk}A_{jl}-A_{kl}A_{jk}+A_{jl}A_{kl}=0, \tag{209}$$

with distinct indices j,k,l. Associated geometrical configurations on the plane are of interest too. For example, at $N=4$ one has four connected KP configurations of the type shown in figure 2. For the triples of points (Φ_1,Φ_2,Φ_3), (Φ_1,Φ_2,Φ_4), (Φ_1,Φ_3,Φ_4), (Φ_2,Φ_3,Φ_4) one has four triples

$$(\Phi_{12},\Phi_{13},\Phi_{23}),(\Phi_{12},\Phi_{14},\Phi_{24}),(\Phi_{13},\Phi_{14},\Phi_{34}),(\Phi_{23},\Phi_{24},\Phi_{34})$$

of collinear points. The latter forms the classical Menelaus configuration (figure 1).

Deformations of such multi-KP configurations are governed by the CS

$$A_{kl}T_lA_{jk}=A_{jk}T_jA_{kl}, \tag{210}$$

$$A_{jk}T_jA_{kl}-A_{jl}T_jA_{kl}-A_{jl}T_lA_{jk}=0. \tag{211}$$

Similar to the case $N=3$, this system is equivalent to the associativity condition (209) and equation (210). Equations (210) imply the existence of the function τ such that

$$A_{jk}=-\frac{\tau_j\tau_k}{\tau\tau_{jk}}. \tag{212}$$

Substitution of this expression into (209) gives

$$\tau_j\tau_{kl}-\tau_k\tau_{jl}+\tau_l\tau_{jk}=0, \qquad j,k,l=1,\ldots,N, \tag{213}$$

with the distinct indices j,k,l. This is the multidimensional version of the Hirota–Miwa equation (192).

One has also the multidimensional versions of relations (197), parametrizations of A_{jk} and B_{jk} given by (201) and (202) as well as the gauge equivalence between the multi-Menelaus and multi-KP configurations.

13. Discrete Darboux system and discrete BKP Hirota–Miwa equation

Now we will turn back to the algebra (150) with unite element. The table of multiplication for the three-dimensional irreducible subalgebra is given by (163)–(165) and the distinguished gauge is defined by relations (170)–(172). Geometrically the triples of structure constants (A, B, L), (C, D, M), (E, G, N) are the barycentric (or normalized) coordinates of the points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ in the plane of the given triangles with the vertices in points (Φ_1, Φ_2, Φ), (Φ_1, Φ_3, Φ), (Φ_2, Φ_3, Φ), respectively.

Equations (166) from the CS imply that there exist three functions U, V, W such that

$$A = \frac{U_2}{U}, \qquad B = \frac{V_1}{V}, \qquad C = \frac{U_3}{U}, \qquad D = \frac{W_1}{W}, \qquad E = \frac{V_3}{V}, \qquad G = \frac{W_2}{W}. \tag{214}$$

In terms of the functions H^1, H^2, H^3 defined by

$$U = H_1^1, \qquad V = H_2^2, \qquad W = H_3^3 \tag{215}$$

the CS (161)–(164) under the constraints (170)–(172) takes the form

$$H^j_{lk} - \frac{H^k_{kl}}{H^k_k} H^j_k - \frac{H^l_{kl}}{H^l_l} H^j_l + \frac{H^k_{lk}}{H^k_k} H^j + \frac{H^l_{lk}}{H^l_l} H^j - H^j = 0 \tag{216}$$

or equivalently

$$\Delta_l \Delta_k H^j - \frac{\Delta_l H^k_k}{H^k_k} \cdot \Delta_k H^j - \frac{\Delta_k H^l_l}{H^l_l} \cdot \Delta_l H^j = 0 \tag{217}$$

where indices $j, k, l = 1, 2, 3$ are all distinct. It is the well-known discrete Darboux system which was first derived in [74]. It describes the deformations of barycentric coordinates discussed above.

For the general N-dimensional case (150) with the constraints

$$A_{jk} + B_{jk} + C_{jk} = 1, \qquad j \neq k, \tag{218}$$

one has

$$A_{jk} = \frac{H^k_{kj}}{H^k_k}, \qquad B_{jk} = \frac{H^j_{kj}}{H^j_j}, \qquad C_{jk} = 1 - \frac{H^k_{kj}}{H^k_k} - \frac{H^j_{kj}}{H^j_j} \tag{219}$$

and the CS is given by N-dimensional systems (216) or (217).

So, the discrete Darboux system governs discrete deformations of the structure constants for the distinguished elements of the gauge equivalency orbits.

The discrete Darboux system (217) is an important system in discrete geometry, in particular, in the theory of quadrilateral lattices (see e.g. [94, 95]). The above result shows that the theory of quadrilateral lattices and discrete deformations of the algebras (150) are strongly interconnected. This connection provides us with some new interpretations of notions used in discrete geometry. For instance, the notion of consistency around the cube discussed in [70, 96] is strictly related to the condition of associativity

$$\mathbf{P}_1(\mathbf{P}_2\mathbf{P}_3)\,|\Psi\rangle = \mathbf{P}_2(\mathbf{P}_1\mathbf{P}_3)\,|\Psi\rangle = \mathbf{P}_3(\mathbf{P}_1\mathbf{P}_2)\,|\Psi\rangle \tag{220}$$

for the three-dimensional algebra of the type (150). Multidimensional consistency (see e.g. [43]) is the associativity condition (152), i.e.

$$p_l(p_j p_k)\,|\Psi\rangle = p_j(p_k p_l)\,|\Psi\rangle\,, \qquad j, k, l = 1, \ldots, N, \tag{221}$$

with distinct indices j, k, l. Due to the connection between the multidimensional consistency and some incidence theorems demonstrated in [97] the latter are also related to associativity conditions.

The discrete Darboux system (217) governs deformations of generic structure constants for the algebra (150) modulo gauge transformations. Deformations of the constrained structure constants are of interest too. One of the examples is provided by the deformations of the three-dimensional algebra (163)–(165) with the additional constraint

$$L = M = N = 1, \tag{222}$$

i.e.

$$A + B = 0, \qquad C + D = 0, \qquad E + G = 0. \tag{223}$$

In this case, the CS (166)–(169) becomes

$$\frac{A_3}{A} = \frac{C_2}{C} = \frac{E_1}{E}, \tag{224}$$

$$A_3 C + E_1 C - E_1 A - 1 = 0. \tag{225}$$

Equations (224) again lead to expressions (191) for A, C, E and equation (225) takes the form

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} - \tau\tau_{123} = 0. \tag{226}$$

This equation is the Hirota–Miwa bilinear discrete equation for the KP hierarchy of B type (BKP hierarchy) [92]. So, the Hirota–Miwa equation (226) together with formulae (191) describes discrete deformations of the algebra (163)–(165) under the constraints (222), (223). In contrast to the KP case, these deformations are not isoassociative.

Equations (177) in the BKP case have the form

$$\Phi_{12} = A(\Phi_1 - \Phi_2) + \Phi, \qquad \Phi_{13} = C(\Phi_1 - \Phi_3) + \Phi, \qquad \Phi_{23} = E(\Phi_2 - \Phi_3) + \Phi. \tag{227}$$

Hence,

$$A = \frac{\Phi_{12} - \Phi}{\Phi_1 - \Phi_2}, \qquad C = \frac{\Phi_{13} - \Phi}{\Phi_1 - \Phi_3}, \qquad E = \frac{\Phi_{23} - \Phi}{\Phi_2 - \Phi_3}. \tag{228}$$

Substituting these expressions into (224), one gets

$$\frac{(\Phi_1 - \Phi_2)(\Phi_3 - \Phi_{123})}{(\Phi_2 - \Phi_3)(\Phi_{123} - \Phi_1)} = \frac{(\Phi_{23} - \Phi_{13})(\Phi_{12} - \Phi)}{(\Phi_{13} - \Phi_{12})(\Phi - \Phi_{23})}. \tag{229}$$

This is the NSF for the Schwarzian BKP hierarchy [98]. Geometrically, this 8-point relation has the meaning of the characteristic equation for two reciprocal triangles on the plane [49]. Considered as the discrete equation, it defines a lattice on the plane consisting of reciprocal triangles. For more details including connection with the works of Maxwell see [49].

Finally, we note that the BKP lattice is gauge equivalent to a particular Darboux lattice. Indeed, performing the gauge transformation $\Phi = g\widetilde{\Phi}$ in equations (227) with $g = \frac{\Phi}{\widehat{\Phi}}$ where $\widehat{\Phi}$ is a solution of (227), one obtains the system of linear equations for $\widetilde{\Phi}$ with coefficients obeying relations (170), (172). The Menelaus lattice is, obviously, the reduction of the Darboux lattice with $L = M = N = 0$. Thus, the discrete Darboux system (217) plays the central role in the theory of discrete deformations of the algebras of the type (150).

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